Exercise 10.1 (Baire categories):

Let \((V, \mathcal{P}), (W, \mathcal{Q})\) be Fréchet spaces over \(\mathbb{F}\). Suppose \(T : V \to W\) is a closed linear mapping. Prove that the following statements are equivalent:

(i) \(rg(T)\) is of category II in \(W\),

(ii) \(T\) is surjective.

Exercise 10.2 (Fourier transform):

For \(f \in L^1([-\pi, \pi], \frac{1}{2\pi} dt)\) define \(c_k(f) = \hat{f}(k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-ikt} \, dt, \; k \in \mathbb{Z}\). Suppose there is a norm \(\| \cdot \|\) on \(L^1([-\pi, \pi], \frac{1}{2\pi} dt)\) such that

(i) \((L^1([-\pi, \pi], \frac{1}{2\pi} dt), \| \cdot \|)\) is a Banach space,

(ii) \(c_k\) is a continuous linear functional on \((L^1([-\pi, \pi], \frac{1}{2\pi} dt), \| \cdot \|)\) for every \(k \in \mathbb{Z}\).

Prove that \(\| \cdot \|\) and \(\| \cdot \|_1\) are equivalent norms on \((L^1([-\pi, \pi], \frac{1}{2\pi} dt), \| \cdot \|)\).

Exercise 10.3 (Multiplication operators):

a) Let \(1 \leq p, q < \infty\) and let \(a = (\alpha_n)_{n \in \mathbb{N}}\) be a sequence s.t. for all \(x = (\xi_n)_{n \in \mathbb{N}} \in \ell^p\) we have \(y = (\alpha_n \xi_n)_{n \in \mathbb{N}} \in \ell^q\). Prove that the multiplication operator \(M_a : \ell^p \to \ell^q\), \(M_a x = y\) is linear and continuous.

b) Let \(1 \leq q < p < \infty\) and let \(a = (\alpha_n)_{n \in \mathbb{N}}\) be a sequence s.t. for all \(x = (\xi_n)_{n \in \mathbb{N}} \in \ell^p\) we have \(y = (\alpha_n \xi_n)_{n \in \mathbb{N}} \in \ell^q\). Define \(M_a\) as in a). Prove that \(M_a : \ell^p \to \ell^q\) is linear and continuous, \(a \in \ell^r\) and \(\|M_a\| = \|a\|_r\), where \(1/p + 1/r = 1/q\).

[Hint: a) Use the closed graph theorem. b) Let \(\xi_n \in \mathbb{F}\) be so that \(|\xi_n| = |\alpha_n|^{(r-q)/q}\). Consider the action of \(M_a\) on \(x_k = (\xi_1, \xi_2, \ldots, \xi_k, 0, 0, \ldots)\) and estimate the norm \(\|M_a x_k\|_q\) to get \((\sum_{j=1}^k |\alpha_j|^r)^{1/r} \leq \|M_a\|\). Show the upper estimate by using Hölder’s inequality.]

Exercise 10.4 (Lebesgue space):

Let \(L^p = L^p([0, 1], dt)\), where \(dt\) denotes the Lebesgue measure on the unit interval \([0, 1]\). Suppose \(E \subset L^p\), \(1 \leq p < \infty\), is a closed linear subspace with \(E \subset L^\infty\). Prove the following statements:

(i) The inclusion operator \(\iota : E \to L^\infty\) is continuous.

(ii) There is a constant \(M > 0\) such that \(\|f\|_\infty \leq M \|f\|_2\) for \(f \in E\).

[Hint: For (ii) use (i) and the fact that \(\|f\|_p^p = \int_0^1 |f(t)|^p |f(t)|^{p-2} dt\) for \(p \in ]2, \infty[\).]
Exercise 10.5 (Closed subspace of continuous functions):
Let \( E \subset C([0, 1]) \) be a closed linear subspace consisting only of \( C^1([0, 1]) \) functions. Prove that \( E \) is finite-dimensional.

[Hint: Apply the closed graph theorem to show that \( D : E \to C([0, 1]), Df = f' \) is bounded and obtain \( \| f' \|_\infty \leq M \) for all \( f \in \overline{B}_E := \{ f \in E : \| f \| \leq 1 \} \). Make use of the Arzela-Ascoli Theorem to show that \( \overline{B}_E \) is compact. Use the fact that the closed unit norm ball \( \overline{B}(0, 1) = \{ x \in V : \| x \| \leq 1 \} \) in a normed vector space \( (V, \| \cdot \|) \) is compact if and only if \( V \) has finite dimension.]

Exercise 10.6 (Schauder basis):
Let \( (V, \| \cdot \|) \) be a Banach space over \( F \). A sequence \( (x_k)_{k \in \mathbb{N}} \subset V \) is called Schauder basis of \( V \) if for every \( x \in V \) there is a uniquely determined sequence \( \xi_k(x) \in F \) such that

\[
x = \sum_{k=1}^{\infty} \xi_k(x) x_k.
\]

a) Assume \( (x_k)_{k \in \mathbb{N}} \) is a Schauder basis of \( V \) and define a sequence of linear operators by

\[
P_n : V \to V, \quad P_n x = \sum_{k=1}^{n} \xi_k(x) x_k, \quad n \in \mathbb{N}.
\]

(i) Check that \( P_n \) is a linear projection operator onto \( \text{span}\{x_k : 1 \leq k \leq n\} \).

(ii) For \( x \in V \) let

\[
\| x \| = \sup_n \| P_n x \|.
\]

Show that \( \| \cdot \| \) is a norm on \( V \) which is equivalent to \( \| \cdot \| \) and that \( (V, \| \cdot \|) \) is a Banach space.

(iii) Prove that \( P_n \) is continuous and uniformly bounded, i.e. \( \sup_n \| P_n \| < \infty \).

(iv) Prove that the coefficient functionals \( \xi_k : V \to F, x \mapsto \xi_k(x) \) are linear, continuous, and fulfil the biorthogonality relation \( \xi_k(x_j) = \delta_{k,j} \).

b) Prove that a sequence of vectors \( (x_n)_{n \in \mathbb{N}} \subset V \) is a Schauder basis iff the following conditions hold.

(i) \( x_n \neq 0 \) for all \( n \in \mathbb{N} \).

(iii) There is a constant \( K > 0 \) so that for every choice \( (\xi_n)_{n \in \mathbb{N}} \subset F \) and all \( n, m \in \mathbb{N}, n < m \),

\[
\left\| \sum_{k=1}^{n} \xi_k x_k \right\| \leq K \left\| \sum_{k=1}^{m} \xi_k x_k \right\|.
\]

(iii) \( \text{c.f.} \| \cdot \|\text{span}\{x_n : n \in \mathbb{N}\} = V \).

(c) Define a sequence of functions \( (\phi_n)_{n \in \mathbb{N}} \) for \( k \in \mathbb{N}, \ l = 1, 2, \ldots, 2^k \) by

\[
\phi_{2^k+l}(t) := \begin{cases} 
1, & t \in \left[ (2l - 2) 2^{-k-1}, (2l - 1) 2^{-k-1} \right], \\
-1, & t \in \left[ (2l - 1) 2^{-k-1}, 2l 2^{-k-1} \right], \\
0, & \text{otherwise}.
\end{cases}
\]
Prove by using the result in b) that \((\phi_n)_n\) is a Schauder basis for all \(L^p([0,1])\), \(1 \leq p < \infty\).

**Hint:** Consider the dyadic intervals of the form \([2^{-k}, (l + 1)2^{-k}]\) to show b)(iii). To show b)(ii) consider the functions \(f(t) = \sum_{k=1}^{n} \alpha_k \phi_k(t)\) and \(g(t) = \sum_{k=1}^{n+1} \alpha_k \phi_k(t)\) and analyze the difference in order to obtain \(\|f\|_p \leq \|g\|_p\).

**Remark.**
1) Note that a Schauder basis \((x_n)_n\) is a sequence, i.e. it comes with a special ordering. It is in general not true that if \((x_n)_n\) is a Schauder basis of \(V\) then for any permutation \(\sigma : \mathbb{N} \to \mathbb{N}\) the system \((x_{\sigma(n)})_{n \in \mathbb{N}}\) is also a Schauder basis. If this is the case the Schauder basis is called unconditional.

2) Don’t confuse Schauder basis with Hamel basis (also called algebraic basis). A Hamel basis \(\{e_\lambda : \lambda \in \Lambda\}\) for \(V\) is a set of linearly independent vectors with \(\text{span}\{e_\lambda : \lambda \in \Lambda\} = V\). This means every \(x \in V\) is uniquely representable by a finite linear combination of elements in \(\{e_\lambda : \lambda \in \Lambda\}\). It can be shown using Baire’s Theorem that a Hamel basis for an infinite-dimensional Banach space must be uncountable. Try to do it.

3) The system in c) is called Haar system. Integrating the elements of the Haar system leads to the system

\[
\varphi_1(t) = 1, \quad \varphi_n(t) = \int_0^t \phi_{n-1}(s) \, ds,
\]

which is the famous Schauder system. It is a Schauder basis for the space \(C([0,1])\) (but not a unconditional Schauder basis). Draw some graphs and maybe try to prove the Schauder basis property.

Clearly, every Banach space which has a Schauder basis is separable. So we see from the constructions above that all spaces \(L^p([0,1])\), \(1 \leq p < \infty\) and the space \(C([0,1])\) are separable Banach spaces. However, the space \(L^\infty([0,1])\) is not separable, consequently there does not exist a Schauder basis for this space.

4) There are several questions poping up.

a) Does every separable Banach space has a Schauder basis?

b) Does every infinite dimensional Banach space contain a basic sequence, i.e. a sequence which is a Schauder basis for the closure of its linear span?

c) Does every separable Banach space embed isometrically into a Banach space with a Schauder basis?

Problem a) was open for about 40 years. This problem was finally answered negatively by Per Enflo in 1972, published in 1973 (A counterexample to the approximation problem in Banach spaces. Acta Mathematica 130 (1)). The prize awarded to Per Enflo for this achievement was a live goose. You can find a bit more facts about the history on Wikipedia (search for Per Enflo).

The question b) was answered positively by Stanislaw Mazur in 1933.

Question c) was answered positively by Stefan Banach and Stanislaw Mazur. They showed that every separable Banach space can be isometrically embedded into \(C([0,1])\) (Zur Theorie der linearen Dimensionen. Studia Mathematica 4, 100-112, 1933).

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This sheet will be discussed from Monday, January 9 on.