Functional Analysis

Exercise Sheet 8

Exercise 8.1 (Two integral transforms):

a) For a complex valued function \( f \in L^1(\mathbb{R}) \) define its Fourier transform as

\[
\mathcal{F}f(\xi) = \int_{-\infty}^{\infty} f(x) e^{-i\xi x} \, dx.
\]

Show that \( \mathcal{F} : L^1(\mathbb{R}) \to C_b(\mathbb{R}) \) is a well-defined, linear, and continuous operator and give an estimate for its operator norm.

b) Let \( s > 0 \) and define for a function \( f : [0, \infty[ \to \mathbb{C} \) its Laplace transform as

\[
\mathcal{L}f(s) = \int_{0}^{\infty} f(t) e^{-st} \, dt.
\]

Prove \( \mathcal{L} : L^2(\mathbb{R}_{>0}) \to L^2(\mathbb{R}_{>0}) \) is a well-defined, linear, and continuous operator with \( \|\mathcal{L}\| = \sqrt{\pi} \). Show that \( \mathcal{L} : L^p(\mathbb{R}_{>0}) \to L^p(\mathbb{R}_{>0}) \) is unbounded for all \( 1 \leq p \leq \infty, \ p \neq 2 \).

Exercise 8.2 (Application of Hahn-Banach):

Let \( V \) be a vector space over \( \mathbb{R} \). A mapping \( q : V \to \mathbb{R} \) is called sublinear if for \( \alpha \geq 0, x, y \in V \)

\[
q(\alpha x) = \alpha q(x), \quad q(x + y) \leq q(x) + q(y).
\]

Let \( W \) be a linear subspace of \( V \) and suppose that \( \varphi_0 \in W' \) satisfies \( \varphi_0(x) \leq q(x) \) for all \( x \in W \). Prove that there is a \( \varphi \in V' \) s.t.

\[
\varphi(x) = \varphi_0(x) \quad \text{for all} \quad x \in W, \quad \varphi(x) \leq q(x) \quad \text{for all} \quad x \in V.
\]

Exercise 8.3 (Hahn-Banach and seminorms):

Let \( V \) be a vector space, \( \varphi \in V' \), and let \( p_1, \ldots, p_n \) be seminorms on \( V \) s.t. \( |\varphi(x)| \leq \sum_{k=1}^{n} p_k(x) \) for all \( x \in V \). Prove that there are \( \varphi_1, \ldots, \varphi_n \in V' \) s.t.

\[
\varphi = \sum_{k=1}^{n} \varphi_k, \quad |\varphi_k(x)| \leq p_k(x) \quad \text{for all} \quad x \in V.
\]

[Hint: Consider the product space \( V^n = V \times \cdots \times V \) with a suitable seminorm constructed from the \( p_k \)'s and the space \( D_n = \{(x, \ldots, x) : x \in V\} \). Consider now the functional on \( D_n \) defined as \( \psi(x, \ldots, x) = \varphi_0(x) \). Apply the Hahn-Banach Theorem to \( \psi \) and derive the desired result.]
Exercise 8.4 (Banach limit):
Let $\ell^\infty_\mathbb{R}$ be the linear space of all real-valued bounded sequences over $\mathbb{R}$. For $x = (x_n)_{n \in \mathbb{N}}$ define

$$p(x) = \limsup_{n \to \infty} \frac{x_1 + \cdots + x_n}{n}$$

and let

$$W = \left\{ x \in \ell^\infty_\mathbb{R} : \lim_{n \to \infty} \frac{x_1 + \cdots + x_n}{n} \text{ exists} \right\}.$$ 

Prove the following statements:

(i) $W$ is a linear subspace of $\ell^\infty_\mathbb{R}$ and $\varphi_0(x) = \lim_{n \to \infty} \frac{x_1 + \cdots + x_n}{n}$ is a functional on $W$ with $\varphi_0(x) = p(x)$.

(ii) There is a linear functional $\text{LIM}$ on $\ell^\infty_\mathbb{R}$ with the following properties

(a) $\text{LIM}(x_1, x_2, \ldots) = \text{LIM}(x_2, x_3, \ldots)$, i.e. $\text{LIM}$ is translation invariant $^1$,

(b) $\liminf_{n \to \infty} x_n \leq \text{LIM}(x_1, x_2, \ldots) \leq \limsup_{n \to \infty} x_n$

(c) $\text{LIM}$ is continuous with $\|\text{LIM}\| = 1$.

[Hint: Use exercise 8.2 to extend the functional to $\ell^\infty_\mathbb{R}$.

Exercise 8.5 (Space of null sequences):
Let $c_0 = \{(\alpha_k)_{k \in \mathbb{N}} : \alpha_k \to 0 \text{ for } k \to \infty\}$. Prove that any linear continuous functional $\varphi_0$ on $c_0$ has one and only one extension to a linear continuous functional $\varphi$ on $\ell^\infty$ with $\|\varphi\| = \|\varphi_0\|$.

[Hint: Proceed as follows. 1) Use $c_0^* \cong \ell^1$, i.e. for $\varphi_0 \in c_0^*$ there is $a = (a_n)_{n \in \mathbb{N}} \in \ell^\infty$ s.t. $\varphi_0(x) = \sum_{n=1}^\infty a_n x_n$, $x = (\xi_k)_{k \in \mathbb{N}}$, and construct a norm-preserving extension $\varphi$ of $\varphi_0$. 2) Assume there is another norm-preserving extension $\tilde{\varphi} \in (\ell^\infty)^*$ of $\varphi_0$. For $x \in \ell^\infty$, $\|x\|_\infty \leq 1$ consider $y_n = (\text{sgn}(a_1), \ldots, \text{sgn}(a_n), \xi_{n+1}, \xi_{n+2}, \ldots)$ and deduce $\tilde{\varphi}(y_n) \to \psi(x) + \|\varphi_0\|$, where $\psi = \tilde{\varphi} - \varphi$. 3) Note that for $x \in \ell^\infty$, $\|x\|_\infty \leq 1$ there is $\lambda \in \mathbb{C}$ s.t. $|\psi(x)| = \lambda \psi(x)$.

4) Use this fact to deduce the desired result $\tilde{\varphi} = \varphi$.]

Exercise 8.6 (Space of summable sequences):
Let $W = \{(\alpha_k)_{k \in \mathbb{N}} \in \ell^1 : \alpha_{2k-1} = 0 \text{ for all } k \in \mathbb{N}\}$. Prove that any $\varphi_0 \in W^* \setminus \{0\}$ has infinitely many extensions to $\varphi \in (\ell^1)^*$ with $\|\varphi\| = \|\varphi_0\|$.

[Hint: 1) Use the fact that $(\ell^1)^* \cong \ell^\infty$, i.e. for every $\varphi \in (\ell^1)^*$ there is $a = (a_k)_{k \in \mathbb{N}} \in \ell^\infty$ s.t. $\varphi(x) = \sum_{k=1}^\infty a_k \xi_k$, $x = (\xi_k)_{k \in \mathbb{N}}$, and construct a norm-preserving extension for $\varphi_0$. 2) Prove that $\|\varphi\| = \sup_k |\alpha_{2k}|$. 3) Choose the $\tau_k$’s in the sequence $(\tau_1, \alpha_2, \tau_3, \alpha_4, \ldots)$ in a suitable manner to obtain the result.]

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This sheet will be discussed from Monday, December 12 on.

$^1$LIM is called Banach limit of $x$. 

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