Exercise 5.1 (Inequalities for measurable functions):
Let \((X, \mathcal{A}, \mu)\) be a measure space. For \(p \in [0, \infty] \) define \(\infty^p := \infty, \; \infty^{-p} := 0\). Let \(f : X \to \mathbb{F}^1\) be a measurable function and define
\[
\|f\|_p := \left( \int_X |f(t)|^p \; d\mu(t) \right)^{1/p}, \; p \neq \infty, \\
\|f\|_\infty = \text{ess sup}_{t \in X} |f(t)| := \inf \{c \in [0, \infty] : |f(t)| \leq c \; \mu\text{-a.e.} \}.
\]

a) (Hölder inequality)
Let \(1 \leq p, q \leq \infty\) with \(p^{-1} + q^{-1} = 1\) and let \(f, g : X \to \mathbb{F}\) be measurable functions. Prove
\[
\|fg\|_1 \leq \|f\|_p \|g\|_q.
\]

b) (Minkowski inequality)
Let \(1 \leq p \leq \infty\) and let \(f, g : X \to \mathbb{F}\) be measurable functions. Prove
\[
\|f + g\|_p \leq \|f\|_p + \|g\|_p.
\]
Show that for \(0 < p \leq 1\) we have
\[
\|f + g\|_p^p \leq \|f\|_p^p + \|g\|_p^p \\
\|f + g\|_p \leq 2^{\frac{1}{p} - 1}(\|f\|_p + \|g\|_p).
\]

Exercise 5.2 (\(L^p\) and \(L^p\)):
Let \((X, \mathcal{A}, \mu)\) be a measure space. For \(0 < p \leq \infty\) define
\[
L^p(X, \mu) = \{f : X \to \mathbb{F} : f \mu\text{-measurable} , \|f\|_p < \infty \}.
\]

a) Demonstrate that for \(1 \leq p \leq \infty\) the space \((L^p(X, \mu), \| \cdot \|_p)\) is in general not a normed vector space but the space \((L^p(X, \mu), \| \cdot \|_p)\) is.

b) Show that for \(0 < p < 1\) the space \((L^p(X, \mu), d_p)\) with \(d_p(f, g) = \|f - g\|_p^p\) is a metric space, but \((L^p(X, \mu), \| \cdot \|_p)\) is not a normed space.

\[^1\mathbb{F} = \mathbb{R} = \mathbb{R} \cup \{-\infty, \infty\}, \text{ or } \mathbb{F} = \mathbb{C}\]
Exercise 5.3 (Banach space of $k$-times continuously differentiable functions):
For $n \in \mathbb{N}$ define
\[ C^n([0,1]) = \{ f : [0,1] \to \mathbb{R} : f^{(k)} \in C([0,1]) \; \forall \; 0 \leq k \leq n \} \]
For $f \in C^n([0,1])$ let
\[ \|f\|_n := \sum_{k=0}^{n} \|f^{(k)}\|_{\infty}. \]
Prove that $(C^n([0,1]), \| \cdot \|_n)$ is a Banach space over $\mathbb{R}$.

Exercise 5.4 (Space of smooth functions):
Let
\[ C^\infty([0,1]) := \bigcap_{n=0}^{\infty} C^n([0,1]) \quad \text{and} \quad p_n(f) := \|f\|_n, \; f \in C^\infty([0,1]), \; n \in \mathbb{N}. \]
Prove that $(C^\infty([0,1]), P)$ with $P = \{ p_n : n \in \mathbb{N} \}$ is a seminormed vector space.

Define a metric on $C^\infty([0,1])$ by
\[ d(f, g) = \sum_{n=0}^{\infty} \frac{1}{2^n} \frac{p_n(f-g)}{1 + p_n(f-g)}. \]
Show that $(C^\infty([0,1]), d)$ is a complete metric space.

This sheet will be discussed from Monday, November 21 on.