Exercise 2.1 (Open ball topology):
Let \((X, d)\) be a metric space.

a) We call \(U \subseteq X\) a neighbourhood of a point \(x \in X\) if there is \(r > 0\) such that \(B(x, r) \subset U\).

Show that the neighbourhood system
\[
\mathcal{U}_x = \{U \subseteq X : U \text{ neighbourhood of } x\}
\]
satisfies the conditions (N1)-(N4) of Proposition 4.6. and hence by Theorem 4.7 defines a topology \(T_d\) on \(X\).

b) A metric \(d'\) is called equivalent to a metric \(d\) if there are constants \(c_0, c_1 > 0\) such that
\[
c_0 d(x, y) \leq d'(x, y) \leq c_1 d(x, y)
\]
for all \(x, y \in X\). Show that equivalent metrics induce the same topology on \(X\).

Exercise 2.2 (Space of sequences):
Let \((X_n, d_n), n \in \mathbb{N}\) be a family of metric spaces and let
\[
X = \prod_{n=1}^{\infty} X_n = \{ (x_n)_{n \in \mathbb{N}} : x_j \in X_j \}.
\]

Moreover, let \((\eta_n)_{n \in \mathbb{N}} \subseteq ]0, \infty[\) be a sequence with \(\sum_{n=1}^{\infty} \eta_n < \infty\) and define for elements \(x = (x_n)_{n \in \mathbb{N}}, y = (y_n)_{n \in \mathbb{N}}\) in \(X\)
\[
d(x, y) = \sum_{n=1}^{\infty} \eta_n \frac{d_n(x_n, y_n)}{1 + d_n(x_n, y_n)}
\]
Show that \(d\) is a metric on \(X\).
Exercise 2.3 (Continuity in metric spaces):
Let \((X,d)\) and \((\tilde{X},\tilde{d})\) be metric spaces and \(f: X \to \tilde{X}\) be a map. Show that \(f\) is continuous at \(x_0 \in X\) if for every sequence \((x_n)_{n \in \mathbb{N}} \subset X\) with \(x_n \to x_0 \ (n \to \infty)\) we have \(f(x_n) \to f(x_0)\).

Exercise 2.4 (Cauchy sequences and completeness):
Let \((X,d)\) be a metric space.
A sequence \((x_n)_{n \in \mathbb{N}}\) in \(X\) is called convergent to \(x \in X\) if for every \(\varepsilon > 0\) there is a \(n_0 \in \mathbb{N}\) such that \(d(x_n, x) < \varepsilon\) for all \(n \geq n_0\).
A sequence \((x_n)_{n \in \mathbb{N}}\) in \(X\) is called Cauchy sequence if for every \(\varepsilon > 0\) there is \(n_0 \in \mathbb{N}\) such that \(d(x_n, x_m) < \varepsilon\) for all \(n, m \geq n_0\).

(i) Demonstrate that a convergent sequence is a Cauchy sequence and that the converse is not necessarily true.

(ii) The metric space \((X,d)\) is called complete if every Cauchy sequence is convergent. Show the following statements

   a) If \(d'\) is a metric equivalent to \(d\) then \((X,d)\) is complete iff \((X,d')\) is complete.

   b) If \(M \subset X\) and \((M,d)\) is complete then \(M\) is closed.

   c) Let \(X\) be complete. If \(M \subset X\) is closed then \((M,d)\) is complete.