Hence there is $\beta_0 \in I$ s.t. for all $\beta > \beta_0$

\[
\left| \frac{I_x}{I_x^k} - \left( \frac{I_x}{I_x^k} + \frac{I_y}{I_y^k} \right) \right| = \left| \frac{I_x}{I_x^k} - \frac{I_x}{I_x^k} (x+y) \right| + \left| \frac{I_x}{I_x^k} - \frac{I_y}{I_y^k} \right| + \left| \frac{I_y}{I_y^k} - \frac{I_y}{I_y^k} (y) \right|
\leq \frac{\varepsilon_1}{2} + \frac{\varepsilon_2}{3} + \frac{\varepsilon_3}{3} = \varepsilon.
\]

Since $\varepsilon > 0$ can be chosen as small as we like we get

\[
\frac{I_x}{I_x^k} = \frac{I_x}{I_x} + \frac{I_y}{I_y^k} \quad x, y \in V.
\]

In an analogous manner one proves

\[
\frac{I_x}{I_x} = \frac{1}{I_x^k} \quad x \in F, x \in V
\]

Thus $x^k(x) := \frac{I_x}{I_x}$ is an element of $V'$.

But

\[
|x^k(x)| = \left| \frac{I_x}{I_x} \right| \leq \|x\|
\]

because $\frac{I_x}{I_x} \in \frac{I_x}{I_x} \cdot v^k_m$. Consequently $\frac{I_x}{I_x} = \frac{I_x}{I_x} \cdot v^k_m$.

This gives compactness of $\frac{I_x}{I_x}$ and since $T$ is a homeomorphism it follows that $\frac{I_x}{I_x}$ is $\frac{I_x}{I_x}$-compact.

Remark: The set $\mathcal{B}^k = \{ x^k \in V^k : \|x^k\| \leq \varepsilon \}$ is closed in $(V^k, \|\|)$.

However, this does not necessarily hold for any norm closed and norm bounded set in $V^k$. For example, if $V$ is not reflexive then for $B = \{ x \in V : \|x\| \leq \varepsilon \}$ we have

\[
B = \bigcap_{\|\|} \mathcal{B}^k = \bigcap_{\|\|} \{ x^k \in V^k : \|x^k\| \leq \varepsilon \}
\]

By Theorem 9.1 $f(B)$ is closed in $(V^k, \|\|)$.

But $f(B)$ is not closed in $(V^k, \|\|)$ since by Theorem 9.18 $f(B)$ is dense in $B^k$. 
We have the following surprising consequence of the Banach-Alaoglu Theorem.

**Proposition 9.28**

Let \((V, \|\cdot\|)\) be an infinite-dimensional TVS over \(F\), and let \(B^* = \{x^* \in V^* : \|x^*\| \leq 1\}\).

If \(0 \in \text{int}_{\varphi_w}(B^*)\), then \(\text{int}_{\varphi_w}(B^*) = \emptyset\).

For the proof we need the following result (without proof).

**Lemma 9.29**

Let \((V, \mathcal{J})\) be a TVS over \(F\). Then the following are equivalent:

(i) \((V, \mathcal{J})\) has finite dimension.

(ii) there is a compact set \(B \subseteq V\) with \(\text{int}_{\varphi_w} B \neq \emptyset\) and \(0 \in B\).

**Proof:** (Proposition 9.28)

Assume \(\text{int}_{\varphi_w} B^* \neq \emptyset\). Then \(B^*\) is norm bounded and weak* closed. By the Banach-Alaoglu Theorem, it is weak* compact, and moreover \(0 \in B^*\). Then by Lemma 9.29, \(V^*\) and hence \(V\) is finite dimensional. \(\Box\)

The proof of the Banach-Alaoglu-Th. needed the assumption that \(V\) is a Banach space only for the \((i) \Rightarrow (ii)\) part. So we get the following Corollary.

**Corollary 9.30**

Let \((V, \|\cdot\|)\) be a TVS over \(F\). If \(M \subseteq V^*\) is weak* closed and norm bounded, then \(M\) is weak* compact.
Corollary 9.30.

Let \((V, \|\cdot\|)\) be a reflexive normed vector space and let \(M \subseteq V\).

Then the following are equivalent:
(a) \(M\) is compact in \((V, \|\cdot\|)\).
(b) \(M\) is \(\overline{\text{rel.}}\)-closed and norm bounded in \((V, \|\cdot\|)\).


Corollary 9.31.

Let \((V, \|\cdot\|)\) be a normed vector space. If \((x^*_\alpha) \subseteq V^*\) is a net with \(\sup_\alpha \|x^*_\alpha\| \leq C\), then there is a subnet \((x^*_\beta)\) of \((x^*_\alpha)\) which converges to \((V, \|\cdot\|_{\text{weak}})\) to a point
\(x^* \in B^* = \{x^* \in V^*, \|x^*\| \leq C\}.

Proof. Apply the corollary 9.30 and the fact that in a topological space, a set is compact if every net contains a convergent subnet.

Reflectivity revisited

Using the Banach-Alaoglu result we can derive a remarkable characterization of reflectivity of a Banach space.

**Theorem 9.33**

Let \((V, \| \cdot \|)\) be a Banach space over \(\mathbb{F}\). Then the following statements are equivalent:

1. \((V, \| \cdot \|)\) is reflexive
2. \(B = \{ x \in V : \| x \| \leq 1 \}\) is compact in \((V, \| \cdot \|)\)

**Proof**

1. \(\Rightarrow\) (2): Assume \(B\) is weakly compact.

   Claim: \(j : (V, \omega) \to (V', \omega_{\omega'})\) is a homeomorphism, where \((V', \omega_{\omega'})\) is the linear subspace \(V' \subset V''\) with the relative topology induced by \(\omega_{\omega'}\).

   Clearly, \(j : V \to j(V)\) is bijective. A net \(x_d\) converges weakly to \(x \in V\) iff \(x^*(x) \to x^*(x)\), \(x^* \in V^*\) which is equivalent to \(\langle j_x, x_d \rangle \to \langle j_x, x \rangle\), \(x^* \in V^*\). This in turn is equivalent to \(j(x) \to j(x)\) in weak*.

   And since this is all equivalent we see \(j\) is a homeomorphism from \((V, \omega)\) to \((j(V), \omega_{\omega'})\).

Since \(B\) is weakly compact and \(j\) is a homeomorphism, \(j(B)\) is compact in \((V', \omega_{\omega'})\) and hence \((V', \omega_{\omega'})\) is Hausdorff and therefore \(j(B)\) is closed in \((V, \omega)\).