A sequence \( (x_n)_{n=1}^\infty \) is a map \( \mathbb{N} \to X, \; n \mapsto x_n \)
denoted by \( (x_n)_{n=1}^\infty \).

This concept needs to be generalized when we work on
general topological spaces. We give a motivation.

Let \( (X, d) \) be a metric space with the corresponding topology \( \mathcal{T} \).

Let \( A \subset X \). Then \( x \in \overline{A} \) means that \( x \) is a sequence \( (x_n)_{n=1}^\infty \subset A \)
such that \( x_n \to x \) for \( n \to \infty \) (convergence).

This means for every \( \varepsilon > 0 \) there is \( N \in \mathbb{N} \) s.t. \( d(x_n, x) < \varepsilon \)
whenever \( n > N \).

The following situation shows that this relation is no longer
ture in general for arbitrary top spaces.

Consider \( X = [0,1] \) and let \( \mathcal{J} \) the co-countable topology 
on \( X \). Let \( A = (0,1) \). The set \( A \) is not closed
since \( A = \{ \emptyset \}^c = X \setminus \{1\} \) and \( \{1\} \notin \mathcal{J} \).

\( \overline{A} \) is closed and since \( A \subset \overline{A} \) and \( \overline{A} = \overline{A} \cap \mathcal{J} \).

Suppose \( (x_n)_{n=1}^\infty \subset A \) is any sequence. Let \( B = \{ x_n, x_0, \ldots \} \)
Since \( B \) is countable \( B^c \) is open and \( x \in B^c \).

So, \( B^c \) is a neighborhood of \( x \) which contains no element of
the sequence \( (x_n)_{n=1}^\infty \).

That means no sequence can converge to \( x \).

This example shows that in a topological space there
may be to many neighborhoods of a point \( x \) s.t. a sequence
can populate all of them.
It is therefore necessary to generalize the concept of sequences. This generalization is the concept of nets.

**Definition 4.17**

A partially ordered set (Poset) is a set $I \neq \emptyset$ with a relation $\leq$ s.t.

1. $x \leq x$ for all $x \in I$
2. $x \leq y$ and $y \leq x$ implies $x = y$.
3. $x \leq y$ and $y \leq z$ implies $x \leq z$.

Note that it can happen that elements from $I$ are not comparable, i.e., neither $x \leq y$ nor $y \leq x$ holds. (Ex 3)

A **directed set** is a Poset $I$ s.t. for any $x, y \in I$ there is $z \in I$ s.t. $x \leq z$ and $y \leq z$.

We define now the concept of a net in a top space.

**Definition 4.18.**

A net in a top space $(X, T)$ is a map from a directed set $I$ into $X$.

It is denoted by $(x_i)_{i \in I}$.

Let $(x_i)_{i \in I}$ be a net in $(X, T)$, then $(x_i)_{i \in I}$ is called eventually in $A$ if there is $i \in I$ s.t. $x_i \in A$ for all $i \geq i$. The net $(x_i)_{i \in I}$ converges to $x \in X$ if for any neighborhood $U$ of $x$, $(x_i)_{i \in I}$ is eventually in $U$. (denoted by $x_i \to x$)
A net \((x_k)_{k \in I}\) is called a Cauchy net if for each neighborhood \(U \subseteq X\) there is \(k_0 \in I\) s.t. \(x_k, x_\beta \in U\) for all \(k, \beta \geq k_0\).

The space \((X, T)\) is called complete if every Cauchy net in \((X, T)\) converges.

The space \((X, T)\) is called sequentially complete if every Cauchy sequence in \((X, T)\) converges.

We provide some examples of directed sets.

**Example 4.19**

(a) Clearly, \(I = \mathbb{N}\) (\(\mathbb{Z}, \mathbb{R}\)) with the usual order \(\leq\) is a directed set.

(b) Consider \(I = \mathbb{N}\) with \(\leq\) defined as follows:

\[2m \leq 2n, \quad 2m - 1 \leq 2n - 1 \quad \text{for } m, n \in \mathbb{N}, \quad 2n - 1 \leq 2m \quad \text{for all } m, n \in \mathbb{N} \]

That means for all even and all odd numbers the usual order is preserved, but every odd number is smaller than every even number. Hence, for example:

\[13 \leq 17, \quad 20 \leq 22 \quad \text{but} \quad 27 \not\leq 4.\]

(c) Let \(X \neq \emptyset\) be a set, and let \(I = \mathcal{P}(X)\). Then \(I\) becomes a directed set with \(\leq\) defined by set inclusion, i.e. \(A, B \in I\), then

\[A \leq B \iff A \subseteq B\]
(d) Let \((X,τ)\) be a topological space and \(x \in X\).

Then the neighborhood system \(U_x\) becomes directed set by \(\leq\) defined as:

\[ U \leq V \iff V \subseteq U, \quad U, V \in U_x \]

(ordering by reverse set inclusion)

Using (d) in the last example we can characterize convergence as follows.

**Proposition 4.80**

Let \((X,τ)\) be a topological space, \(x \in X\), and \(U_x\) the neighborhood system at \(x\). Moreover, let \(x \in U\) for every \(U \in U_x\).

Then the net \((x_\alpha)_{\alpha \in \Lambda}\) converges to \(x\), where \(U_x\) is ordered by reverse set inclusion.

**Proof.** It is obvious from Def. 4.18 and Example 4.49 (d).

For a sequence \((x_n)_{n \in \mathbb{N}}\) in a top space \((X,τ)\) it is clear what a subsequence is. We just take \(\{n_1, n_2, n_3, \ldots\} \subseteq \mathbb{N}\) and consider \((x_{n_k})_{k \in \mathbb{N}}\). If we consider a net \((x_\alpha)_{\alpha \in \Lambda}\) the construction of a subnet \((x_\beta)_{\beta \in \Gamma}\) is by no means so obvious.

**Definition 4.21.**

Let \(I, J\) be two directed sets. A map \(F : I \to J\) is called cofinal if for any \(x \in I\) there is \(\beta \in J\) s.t. \(F(\beta) \geq x\) whenever \(\beta \geq \beta'\).
If \((x_n)_{n \in I} \subseteq X\) is a net and \(F : I \to I\) is a cofinal map. Then \((x_{F(n)})_{n \in \mathbb{N}}\) is called a subnet of \((x_n)_{n \in I}\).

We consider some examples.

**Example 4.22**

(a) If \(I = \mathbb{N}\) with usual ordering and \(F : \mathbb{N} \to \mathbb{N}\) is any increasing map, then clearly \((x_{F(n)})_{n \in \mathbb{N}}\) is a subsequence resp. subnet of \((x_n)_{n \in \mathbb{N}}\).

(b) Let \(I = \mathbb{N}\) equipped with the usual ordering and

\[ F : \mathbb{N} \to \mathbb{N}, \quad F(n) = 3n. \]

This map is cofinal.

Let \(n \in \mathbb{N}\) and set \(\beta' = 2n\). For \(\beta \gg \beta'\), \(\beta\) must be even and \(\beta \geq \beta'\). Then

\[ F(\beta) = 3\beta \gg \beta \gg \beta' = 2\beta \gg \beta \]

Hence \(F\) is cofinal.

Now let \((x_n)_{n \in \mathbb{N}} \subseteq X\). Then \((x_{F(n)})_{n \in \mathbb{N}} = (x_{3n})_{n \in \mathbb{N}}\) is a subnet of \((x_n)_{n \in \mathbb{N}}\).

However, \((x_{F(n)})_{n \in \mathbb{N}}\) is not a subsequence since the ordering in \(I = \mathbb{N}\) is not the ordinary one.

Indeed, \(F(n) = 3n\), \(x_{3n} = 0\), \(x_{3n+1} = \frac{1}{n+1}\), then

\((x_n)_{n \in \mathbb{N}} = (1, 0, 3, 0, 5, \ldots)\) and \((x_{3n})_{n \in \mathbb{N}} = (3, 0, 9, 0, 27, \ldots)\)

Clearly, \((x_{3n})_{n \in \mathbb{N}}\) does not converge.

However, \((x_{3n})_{n \in \mathbb{N}}\) does converge as we have \(x_{3n} = 0\) for all \(n \gg 2\).
(c) Let $f : R, I = N$ both with usual order $\geq$. Let $F : R \rightarrow N$ be a function with $F(t) \rightarrow \infty$ as $t \rightarrow \infty$. Then $F$ is cofinal.

If $(x_n)_{n \in N}$ is a net resp. sequence, then $(F(x_n))_{n \in N}$ is a subnet but not a subsequence of $(x_n)_{n \in N}$.

Remark:

There are other definitions of a subnet. The definition presented here is due to J. L. Kelley (see J. L. Kelley, General Topology, 1975).