3. Linear Maps

Let $X, Y$ be vector spaces over the same field $F$. A linear map

$$
\phi : X \to Y, \quad x \mapsto \phi(x)
$$

is a linear homomorphism, i.e., we have

$$
\phi(x + x') = \phi(x) + \phi(x'), \quad \text{for all } x, x' \in X
$$
$$
\phi(ax) = a \phi(x), \quad \text{for all } a \in F, x \in X.
$$

The set

$$
\mathcal{L}(X,Y) := \{ \phi : X \to Y : \phi \text{ linear} \}
$$

is itself a $F$-space by

$$
(\phi + \psi)(x) = \phi(x) + \psi(x), \quad \phi, \psi \in \mathcal{L}(X,Y), \quad x \in X
$$
$$
(\alpha \phi)(x) = \alpha \phi(x), \quad \phi \in \mathcal{L}(X,Y), \quad \alpha \in F, \quad x \in X
$$

Let $Z$ be a $F$-space and $\psi \in \mathcal{L}(Y,Z)$, $\phi \in \mathcal{L}(X,Y)$

then

$$
\psi \circ \phi (x) = \psi \phi(x) = \psi(\phi(x))
$$

is the composition of $\psi$ and $\phi$ and we have $\psi \circ \phi \in \mathcal{L}(X,Z)$. Further, if $\psi, \varphi \in \mathcal{L}(X,Y)$, $\eta \in \mathcal{L}(Y,Z)$

$$
\eta(\psi + \varphi) = \eta \psi + \eta \varphi
$$

$$
(\eta \varphi) \psi = \eta \varphi \psi
$$

In the case $X = Y = Z$ we see that $\mathcal{L}(X) = \mathcal{L}(X,X)$ does not only carry a vector space structure but additionally has a multiplicative structure, the composition of linear maps, which interacts by (1) with the vector space structure.
Recall that an algebra \( A \) over a (commutative) field \( F \) is a \( F \)-space with multiplication
\[
\cdot : A \times A \rightarrow A, \quad (a, b) \mapsto ab
\]
such that \((A, \cdot)\) is a ring and
\[
x(ab) = (xa)b = a(xb)
\]

**Proposition 3.1**

Let \( X \) be a \( F \)-space. Then \( \mathcal{L}(X) \) is an algebra with composition.

**Proof.** The statement follows immediately from the facts above. \( \square \)

We collect basic properties of linear maps.

**Proposition 3.2**

Let \( X, Y \) be \( F \)-spaces and \( cp \in \mathcal{L}(X, Y) \). Then
\[
 cp(0) = 0, \quad cp(-x) = -cp(x)
\]

**Proof.**

The next result shows that a linear map is uniquely determined by the image of the basis.

**Proposition 3.3**

Let \( X, Y \) be \( F \)-spaces and \( B \) be a basis of \( X \). If every \( x \in B \) is mapped to a \( y \in Y \), then there is a uniquely determined linear map \( cp : X \rightarrow Y \) such that \( cp(x) = y \) for every \( x \in B \).
Proof. Every vector $x$ can be represented as $x = x_1 x_1 + \ldots + x_n x_n$, with finitely many $x_1, \ldots, x_n \in B$. If $\varphi : X \to Y$ is linear with $\varphi(x) = y$ for $x \in B$ then

$$\varphi(v) = \alpha_1 \varphi(x_1) + \ldots + \alpha_n \varphi(x_n) = \alpha_1 y_1 + \ldots + \alpha_n y_n,$$

i.e. the image of $v$ is uniquely determined by the coefficients $x_1, \ldots, x_n \in F$.

Conversely, $\varphi(v) = \alpha_1 y_1 + \ldots + \alpha_n y_n$ defines a linear map with the property $\varphi(x) = y$ for every $x \in B$.

We moreover have

**Proposition 3.4**

Let $X, Y$ be $F$-spaces and $\varphi \in L(X, Y)$.

1. $\varphi([M]) = [\varphi(M)]$ for every subset $M \subseteq X$.
2. If $V$ is a subspace of $X$, then $\varphi(V)$ is a subspace of $Y$.

Moreover, $\dim \varphi(V) \leq \dim V$.

**Proof.** (1) Let $M \neq \emptyset$. Since $\varphi$ is linear, $\varphi(M)$ consists precisely of all linear combinations of the form

$$\alpha_1 \varphi(x_1) + \ldots + \alpha_n \varphi(x_n) = \varphi(\alpha_1 x_1 + \ldots + \alpha_n x_n).$$

Prop. 3.5 now shows $\varphi([M]) = [\varphi(M)]$.

(2) Let $V \subseteq X$ be a subspace, then $[V] = V$. Hence

$$[\varphi(V)] = [\varphi([V])] = \varphi([V]).$$

Let $B$ be a basis of $V$. Then $[\varphi(B)] = \varphi([B]) = \varphi(V)$.

Hence $\varphi(B)$ generates the subspace $\varphi(V)$. Therefore, $\varphi(B)$ contains a basis of $\varphi(V)$. 


If \(|\mathbf{p}(B)| = \infty\) then \(|B| = \infty\). On the other hand, if
\
|\mathbf{p}(B)| = m < \infty\) then \(|B| > m\). This shows \(\dim(\mathbf{p}(V)) \leq \dim V\).

We introduce the following notation for a linear map \(\mathbf{p} \in \mathcal{L}(X, Y)\).
\[
\text{Ker}(\mathbf{p}) = \{x \in X : \mathbf{p}(x) = \mathbf{0}\} \quad (\text{kernel or null space})
\]
\[
\text{Rng}(\mathbf{p}) = \mathbf{p}(X) = \{y \in Y : \exists x \in X : \mathbf{p}(x) = y\} \quad (\text{range of } \mathbf{p})
\]
\[
\mathbf{p}^{-1}(V) = \{x \in X : \mathbf{p}(x) \in V\} \quad (\text{preimage of } V \in Y)
\]

**Proposition 3.5**

Let \(\mathbf{p} \in \mathcal{L}(X, Y)\). The following statements are equivalent

(i) \(\mathbf{p}\) is surjective

(ii) If \(B\) is a basis of \(X\), then \(|\mathbf{p}(B)| = Y\)

**Proof** Since \(|\mathbf{p}(B)| = \mathbf{p}(|B|) = \mathbf{p}(X)\) we have \(|\mathbf{p}(B)| = Y\)
if and only if \(\mathbf{p}(x) = \mathbf{y} = \mathbf{y}\) i.e. \(\mathbf{p}\) is surjective.

**Proposition 3.6**

Let \(\mathbf{p} \in \mathcal{L}(X, Y)\) and \(V \subseteq Y\) be a subspace. Then \(\mathbf{p}^{-1}(V)\)
is a subspace of \(X\).

**Proof** (obvious)

**Proposition 3.7**

Let \(\mathbf{p} \in \mathcal{L}(X, Y)\). The following statements are equivalent

(i) \(\mathbf{p}\) is injective

(ii) \(\text{Ker}(\mathbf{p}) = \{0\}\)

(iii) If \(B\) is a basis of \(X\), then the vectors \(\mathbf{p}(x), x \in B\)
are pairwise distinct and \(|\mathbf{p}(B)|\) is linearly indep.
Proof. (i) \( \Rightarrow \) (ii) is clear.

(ii) \( \Rightarrow \) (iii) Let \( B \) be a basis of \( x \), \( x_1, \ldots, x_n \in B \).

Since \( \phi \) is injective we see that

\[ \lambda_1 \phi(x_1) + \ldots + \lambda_n \phi(x_n) = \phi(x_1) + \ldots + x_n = 0 \]

implies \( \lambda_1 x_1 + \ldots + \lambda_n x_n = 0 \). But this implies \( \lambda_1 = \ldots = \lambda_n = 0 \) hence the set \( \phi(B) \) is linearly independent.

Now since \( B \) is a basis of \( X \), the vectors \( \psi, \psi' \in X \) have representations

\[ \psi = \alpha_1 x_1 + \ldots + \alpha_n x_n \quad \text{and} \quad \psi' = \alpha'_1 x_1 + \ldots + \alpha'_n x_n \]

If \( \phi(\psi) = \phi(\psi') \) then \( \alpha_1 \phi(x_1) + \ldots + \alpha_n \phi(x_n) = \alpha'_1 \phi(x_1) + \ldots + \alpha'_n \phi(x_n) \)

which implies

\[ (\alpha_1 - \alpha'_1) \phi(x_1) + \ldots + (\alpha_n - \alpha'_n) \phi(x_n) = 0 \]

\( \phi(x_1), \ldots, \phi(x_n) \) are linearly independent. Hence \( \alpha_1 - \alpha'_1 = 0 \), \( \alpha_1 = \alpha'_1 \).

This implies \( \psi = \psi' \) and hence \( \phi \) is injective. \( \square \)

We call a linear map \( \phi \in \mathcal{L}(X,Y) \) an isomorphism if it is bijective (i.e. injective and surjective). In this case the linear spaces are called isomorphic.

In functional analysis it is custom to denote linear maps by capital roman letters and reserve small greek letters for linear maps from \( X \) to \( F \). Often a linear map \( A \in \mathcal{L}(X,Y) \) is called a linear operator. We will henceforth follow this custom.
We consider a special class of linear operators.

Let $X$ be a linear space, $U, V \subseteq X$ subspaces with $X = U \oplus V$.

Every $x \in X$ has a unique decomposition $x = u + v$ with $u \in U, v \in V$.

A linear operator $P \in \mathcal{L}(X)$ with $\ker(P) = U$, $P(x) = u$ as called a projector along the subspace $V$ (along $V$, for short).

Obviously

$P(x) = u$, $\text{ker}(P) = V$, $P^2 = P$ (i.e. $P$ is idempotent).

We have

**Proposition 3.8**

$P \in \mathcal{L}(X)$ is a projector if and only if it is idempotent.

**Proof (Exercise)**