Problem Set 3

1. Suppose the function $f : \mathbb{R} \to \mathbb{R}$ has period $p > 0$. Show that $\int_a^{a+p} f(x) dx$ is independent of $a$.

2. **Fourier Inversion.** Suppose that $f \in L^1(\mathbb{T})$ and that $\{\hat{f}(n)\}_{n \in \mathbb{Z}} \in \ell^1(\mathbb{Z})$. Then

$$f(t) = \sum_{n \in \mathbb{Z}} \hat{f}(n)e^{int}, \text{ a.e.}$$

Therefore, $f$ is almost everywhere equal to a continuous function.

3. **Lipschitz Spaces on $\mathbb{T}$.** Let $0 < \alpha \leq 1$. The subspace $\text{Lip}_\alpha(\mathbb{T})$ of $C(\mathbb{T})$ consists of all functions $f$ which satisfy the Lipschitz condition

$$|f(z_1) - f(z_2)| \leq c|z_1 - z_2|^\alpha, \quad \forall z_1, z_2 \in \mathbb{T},$$

for some $c > 0$.

On the $\mathbb{C}$-vector space $\text{Lip}_\alpha(\mathbb{T})$ a mapping $\| \cdot \| : \text{Lip}_\alpha(\mathbb{T}) \to \mathbb{R}$ is defined by

$$\|f\|_{\text{Lip}_\alpha} := \|f\|_\infty + \sup_{z \in \mathbb{T}, h \neq 0} \frac{|f(z + h) - f(z)|}{|h|^{\alpha}}. \quad (1)$$

(a) Show that $(\cdot)$ defines a norm on $\text{Lip}_\alpha(\mathbb{T})$.

(b) Prove **Bernstein’s Theorem**: Suppose $f \in \text{Lip}_\alpha(\mathbb{T})$ for some $\alpha > \frac{1}{2}$. Then the Fourier series of $f$ is in $\ell^1(\mathbb{Z})$ and

$$\sum_{n \in \mathbb{Z}} |\hat{f}(n)| \leq c_\alpha \|f\|_{\text{Lip}_\alpha},$$

where the positive constant $c_\alpha$ depends only on $\alpha$.

4. For $f \in L^1(\mathbb{T})$ and $h \neq 0$ denote by $\Omega(f, h)$ the **integral modulus of continuity of $f$**, i.e.,

$$\Omega(f, h) := \|f(\cdot + h) - f\|_{L^1}.$$  

Prove that for $0 \neq n \in \mathbb{Z}$,

$$|\hat{f}(n)| \leq \frac{1}{2} \Omega(f, \frac{n}{|n|}).$$