Problem Set 1

1. Let $G$ be a finite Abelian group. A character $\chi$ of $G$ is a group homomorphism $\chi : G \to \mathbb{T}$, i.e.,

$$\chi(gh) = \chi(g)\chi(h), \quad \forall g, h \in G,$$

where $\mathbb{T} \cong \mathbb{R}/2\pi\mathbb{Z}$ is one-dimensional torus. Denote the set of all characters by $\hat{G}$.

Show that the point-wise product $(\chi, \eta) \mapsto \chi \eta$, $\chi \eta(g) = \chi(g)\eta(g)$ makes $\hat{G}$ into an Abelian group. $\hat{G}$ is called the dual group or the Pontryagin dual of $G$.

2. Suppose that $G$ is a cyclic group of order $N$, i.e., $\exists g \in G$ so that $G = \langle 1, g, \ldots, g^{N-1} \rangle$ and $g^N = 1$.

Show that the characters of $G$ are given by

$$\chi_m(g^k) = e^{ikm/N}, \quad k \in \mathbb{Z},$$

for $m \in \{0, 1, \ldots, N - 1\}$. The group $\hat{G}$ is again cyclic of order $N$.

3. Let $G$ be a finite Abelian group and $g \in G$. Assume that $\chi(g) = 1$ for all $\chi \in \hat{G}$.

Show that this implies $g = 1$.

(Hint: First assume that $G$ is cyclic and then use the result that every finite Abelian group is isomorphic to a finite product of cyclic groups.)

4. Suppose $G$ is a finite Abelian group.

Show that there exists a natural isomorphism to the bidual $G \to \hat{\hat{G}}$ given by $g \mapsto \delta_g$, where $\delta_g$ is the point evaluation at $g$:

$$\delta_g : \hat{\hat{G}} \to \mathbb{T},$$

$$\chi \mapsto \chi(g).$$