Exercise 11.1:
Let \( f \) be an extended real-valued function. An element \( v \in \mathbb{R} \) is called a subgradient of \( f \) at \( x_0 \in \mathbb{R} \) if
\[
f(x) \geq f(x_0) + v(x - x_0), \quad \forall x \in \mathbb{R}.
\]
The set of all subgradients of \( f \) at \( x_0 \) is called the subdifferential of \( f \) of \( x_0 \) and denoted by \( \partial f(x_0) \).

Show that for all \( x, v \in \mathbb{R} \),
\[
v \in \partial f(x) \iff f^*(v) = \langle v, x \rangle - f(x).
\]

Exercise 11.2:
Using MATLAB, perform five iterations to find the minimum of \( f(x_1, x_2) := (x_1 - 3)^4 + (x_1 - 3x_2)^2 \)
using the following four methods. In each case, \( \nabla \) and \( H \) denotes the gradient, respectively, the Hessian of \( f \).

1. Newton’s Method:
\[
x_{k+1} = x_k - (H(x_k))^{-1} \nabla f(x_k), \quad k \in \mathbb{N}_0,
\]
with \( x_0 := (0, 0)^\top \).

2. Levenberg-Marquardt’s Method:
\[
x_{k+1} = x_k - (\hat{H}(x_k))^{-1} \nabla f(x_k), \quad k \in \mathbb{N}_0,
\]
where \( \hat{H}(x) := H(x) + \lambda I \), for \( x_0 := (0, 0)^\top \) and \( \lambda := 1.1 \).

3. Broyden-Fletcher-Goldfarb-Shanno’s Method:
\[
x_{k+1} = x_k - (\hat{H}_k)^{-1} \nabla f(x_k), \quad k \in \mathbb{N}_0,
\]
where
\[
\hat{H}_k := \hat{H}_{k-1} + \frac{q_k q_k^\top}{s_k^\top s_k} - \frac{\hat{H}_{k-1} s_k s_k^\top}{s_k^\top s_k} \hat{H}_{k-1}^\top,
\]
\[
s_k := x_k - x_{k-1}, \quad q_k := \nabla f(x_k) - \nabla f(x_{k-1}),
\]
\[
\hat{H}_0 := H(x_0),
\]
for \( x_0 := (0, 0)^\top \).
4. Davidon-Fletcher-Powell’s Method:

\[ x_{k+1} = x_k - \hat{D}_k \nabla f(x_k), \quad k \in \mathbb{N}_0, \]

where

\[ \hat{D}_k := \hat{D}_{k-1} + \frac{s_k q_k^\top}{q_k s_k} - \frac{\hat{D}_{k-1} q_k q_k^\top \hat{D}_{k-1}^\top}{q_k^\top \hat{D}_{k-1} q_k}, \]
\[ s_k := x_k - x_{k-1}, \quad q_k := \nabla f(x_k) - \nabla f(x_{k-1}), \]
\[ \hat{D}_0 := H(x_0)^{-1}, \]

for \( x_0 := (0,0)^\top. \)

**Exercise 11.3:**

Let \( p(z) \) be a complex polynomial and \( \xi \) one of its roots. If the Newton-Raphson method is applied to a starting point \( z_0 \in \mathbb{C} \) it will produce a sequence \( \{z_k\}_{k \in \mathbb{N}_0} \) defined by the equations

\[ z_{k+1} = z_k - \frac{p(z_k)}{p'(z_k)}, \quad k \in \mathbb{N}_0. \]

If \( \lim_{k \to \infty} z_k = \xi \), we say that (the starting point) \( z_0 \) is attracted to \( \xi \). The set of all starting points \( z_0 \) that are attracted to \( \xi \) is called the basin of attraction corresponding to \( \xi \).

Using MATLAB, determine the basins of attraction corresponding to the three roots of the polynomial \( p(z) := z^3 - 1 \). These roots are:

\[ \xi_1 = 1, \quad \xi_2 = \frac{1}{2} (-1 + \sqrt{3} i), \quad \xi_2 = \frac{1}{2} (-1 - \sqrt{3} i). \]

Color those starting points \( z_0 \) that are attracted by \( (\xi_1, \xi_2, \xi_3) \) by (red, green, blue) and display the basins of attraction in one plot. (Hint: You may monitor the first 20 iterates and check whether any of them is within distance 0.25 of a root.)

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This sheet will be discussed on **Monday, July 24 and Tuesday, July 25**.