Foundations of Data Analysis

Exercise 6.1:
Suppose that $X$ is a normally distributed random variable with mean 0 and variance 1. Compute the moment generating function $M_X(\theta) = \mathbb{E}(e^{\theta X})$, $\theta \in \mathbb{R}$.

Exercise 6.2:
Suppose $X_1, \ldots, X_n$ are independent and normally distributed random variables with means $\mu_1, \ldots, \mu_n$ and variances $\sigma_1^2, \ldots, \sigma_n^2$. Show that $Y := X_1 + \cdots + X_n$ is a normal random variable with mean $\sum_{i=1}^n \mu_i$ and variance $\sum_{i=1}^n \sigma_i^2$.

Exercise 6.3:
A sequence of random variables $\{X_n\}_{n \in \mathbb{N}}$ is said to

- converge in mean (square) to a random variable $X$ if
  \[ \mathbb{E}(|X_n - X|^2) \to 0 \quad \text{as } n \to \infty; \]

- converge in probability to a random variable $X$ if
  \[ \forall \epsilon > 0 : \mathbb{P}(|X_n - X| \geq \epsilon) \to 0 \quad \text{as } n \to \infty. \]

Suppose $\{X_n\}_{n \in \mathbb{N}}$ is a sequence of random variables where
\[
\mathbb{P}(X_n = 0) = 1 - \frac{1}{n} \quad \text{and} \quad \mathbb{P}(X_n = n) = \frac{1}{n}, \quad n \in \mathbb{N}.
\]

Show that $\{X_n\}_{n \in \mathbb{N}}$ converges to $X = 0$ in probability but not in mean square. What about the converse? Is it always true that convergence in mean implies convergence in probability? Give a proof of this statement or find a counterexample.

Exercise 6.4:
Suppose $\{X_n\}_{n \in \mathbb{N}}$ is a sequence of independent and identically distributed random variables and $S_n := X_1 + \cdots + X_n$. Assume that each $X_i$ has mean 0 and that all $X_i$ have a common moment generating function $M(\theta)$ which is bounded for all $\theta \in \mathbb{R}$. For any $a > 0$, show that
\[
\mathbb{P}(S_n > an) \leq \left( \frac{M(\theta)}{e^{a\theta}} \right)^n, \quad \theta > 0.
\]
Exercise 6.5:
Draw $n$ points $x_1, \ldots, x_n$ at random from the unit ball in $\mathbb{R}^d$. Show that with probability $1 - \mathcal{O}(\frac{1}{n})$ the following estimates hold.

a) $\|x_i\|_2 \geq 1 - \frac{2\ln n}{d}$, for all $i = 1, \ldots, n$.

b) $|\langle x_i, x_j \rangle| \leq \frac{\sqrt{6\ln n}}{\sqrt{d-1}}$, for all $i, j = 1, \ldots, n$, $i \neq j$.

Exercise 6.6:
Prove the Gaussian Annulus Theorem: For a Gaussian vector in dimension $\mathbb{R}^n$ with mean $\mu = 0 \in \mathbb{R}^n$ and covariance matrix $\Sigma = I \in \mathbb{R}^{n \times n}$ and for any $\beta \leq \sqrt{n}$ all but at most $3e^{-\beta^2}$ of the probability lies within the annulus $\sqrt{n} - \beta \leq \|x\|_2 \leq \sqrt{n} + \beta$.

Exercise 6.7:
A die is thrown 10,000 times and it was noted that “6” came up more than 1834 times. Use Hoeffding’s inequality to determine whether the die is loaded.

This sheet will be discussed on Monday, June 19 and Tuesday, June 20.