Foundations of Data Analysis (MA4800)

Massimo Fornasier

Fakultät für Mathematik
Technische Universität München
massimo.fornasier@ma.tum.de
http://www-m15.ma.tum.de/

Slides of Lecture
July 4 2017
Preliminaries on Linear Algebra (LA)

We give for granted familiarity with the basics of LA taught in standard courses, in particular,

- vector spaces, spans and linear combinations, linear bases, linear maps and matrices, eigenvalues, complex numbers, scalar products, theory of symmetric matrices.

For more details we refer to the lecture notes (in German)
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For more details we refer to the lecture notes (in German) G. Kemper, *Lineare Algebra fuer Informatik*, TUM, 2017.

of the course taught for Computer Scientists at TUM, or any other standard international text of LA.
Matrix Notations

The index sets $I, J, L, \ldots$ are assumed to be finite.

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The set $\mathbb{K}^{I \times J}$ is the vector space of matrices $A \in \mathbb{K}^{I \times J}$, whose entries are denoted by $A_{ij}$, for $i \in I$ and $j \in J$. Vice versa, numbers $a_{ij} \in \mathbb{K}$ may be used to define $A := (a_{ij})_{i \in I, j \in J} \in \mathbb{K}^{I \times J}$.

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If $A \in K^{I \times J}$, the transposed matrix $A^T \in K^{J \times I}$, and $A^T_{ji} := A_{ij}$. A matrix $A \in K^{I \times I}$ is symmetric if $A^T = A$. The Hermitian transposed matrix $A^H \in K^{J \times I}$ coincides with $\overline{A^T}$. If $K = \mathbb{R}$ then clearly $A^H = A^T$. A Hermitian matrix satisfies $A^H = A$.

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Often, we will need to consider the rows $A^{(i)}$ and the columns $A^{(j)}$ of a matrix, defined respectively as vectors $A^{(i)} = (a_{ij})_{j \in J}$ and $A^{(j)} = (a_{ij})_{i \in I} = (A^T)^{(j)}$.

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We assume that the standard matrix-vector and matrix-matrix multiplications are done in the usual way: \((Ax)_i = \sum_{j \in J} a_{ij}x_j\) or \((AB)_{i\ell} = \sum_{j \in J} A_{ij}B_{j\ell}\) as usual, for \(x \in \mathbb{K}^J\), \(A \in \mathbb{K}^{I \times J}\) and \(B \in \mathbb{K}^{J \times L}\).
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The Kronecker symbol is defined by

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\delta_{ij} = \begin{cases} 
1, & \text{if } i = j \in I, \\
0, & \text{otherwise.}
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The symbol \(I = (\delta_{ij})_{j \in I, j \in I}\) is used for the identity matrix. Since matrices and index sets do not appear in the same place, the simultaneous use of the symbol \(I\) do not create confusion (example: \(I \in \mathbb{K}^{I \times I}\)).
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The range of matrix $A \in \mathbb{K}^{I \times J}$ is

$$\text{range}(A) = \{Ax : x \in \mathbb{K}^J\} = \text{span}\{A_{(i)}, i \in I\}.$$ 

Hence, the range of a matrix is a vector space spanned by its columns.
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The Euclidean scalar product in $\mathbb{K}^I$ is given by

$$\langle x, y \rangle = y^H x = \sum_{i \in I} x_i \overline{y}_i,$$  \hspace{1cm} (1)

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It often useful and very important to notice that the matrix-vector $Ax$ and matrix-matrix $AB$ multiplications can also be expressed in terms of scalar products of the rows of $A$ with $x$ and of the rows of $A$ with the columns of $B$, i.e., according to our terminology of

$$(Ax)_i = \langle A(i), \bar{x} \rangle, \quad (AB)_{i\ell} = \langle A(i), \overline{B(\ell)} \rangle.$$
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We often consider sets which are mutually orthogonal. In this case for $X, Y \subseteq \mathbb{K}^I$ we say that $X$ is orthogonal to $Y$ and we write $X \perp Y$ if $\langle x, y \rangle = 0$, for all $x \in X$ and $y \in Y$. 
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When \( X \) and \( Y \) are linear subspaces their orthogonality can be simply checked by showing that they possess bases which are mutually orthogonal. This is a very important principle we will use often.
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A family of vectors $X = \{x_\nu\}_{\nu \in F} \subseteq \mathbb{K}^l$ is orthogonal if the vectors $x_\nu$ are pairwise orthogonal, i.e., $\langle x_\nu, x_{\nu'} \rangle = 0$ for $\nu \neq \nu'$. The family is additionally called orthonormal if $\langle x_\nu, x_\nu \rangle = 1$ for all $\nu \in F$. 
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An orthogonal square matrix $A \in \mathbb{K}^{I \times I}$ is called unitary. Differently from just orthogonal matrices, unitary matrices satisfy

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i.e, $A^H = A^{-1}$ is the inverse matrix of $A$. 
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Assume that the index sets satisfy either $I \subset J$ or $J \subset I$. Then a (rectangular) matrix $A \in \mathbb{K}^{I \times J}$ is diagonal if $A_{ij} = 0$ for all $i \neq j$. 
Matrix Rank

Proposition

Let $A \in \mathbb{K}^{I \times J}$. The following statements are equivalent and may be all used as a definition of matrix rank $r = \text{rank}(A)$.
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- (a) $r = \text{dim range}(A)$;
- (b) $r = \text{dim range}(A^H)$;
- (c) $r$ is the maximal number of linearly independent rows of $A$;
- (d) $r$ is the maximal number of linearly independent columns of $A$;
- (e) $r$ is minimal with the property $A \approx_r \bar{i}$, $a_i \in K^I, b_i \in K^J$;
- (f) $r$ is maximal with the property that there exists an invertible $r \times r$ submatrix of $A$;
- (g) $r$ is the number of positive singular values (soon!).
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A = \sum_{i=1}^{r} a_i b_i^H, \quad \text{where } a_i \in \mathbb{K}^I, b \in \mathbb{K}^J;
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The rank is bounded by the maximal rank, which, for matrices, is always given by \( r_{\text{max}} = \min\{\#I, \#J\} \), and this bound is attained by full-rank matrices.
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We will often work with matrices of bounded rank $r \leq k$. We denote accordingly with $\mathcal{R}_k = \{A \in \mathbb{K}^{I \times J} : \text{range}(A) \leq k\}$ such a set. Notice that this set is not a vector space (exercise!).
Norms

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We recall below the axioms of an abstract norm on $V$: a norm is a map $\| \cdot \| : V \to [0, \infty)$ with the following properties

- $\|v\| = 0$ if and only if $v = 0$;
- $\|\lambda v\| = |\lambda|\|v\|$ for all $v \in V$ and $\lambda \in K$;
- $\|v + w\| \leq \|v\| + \|w\|$ for all $v, w \in V$ (triangle inequality).
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A norm is always continuous as a consequence of the (inverse) triangle inequality:

$$\|\|v\| - \|w\|\| \leq \|v - w\|, \text{ for all } v, w \in V.$$
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A vector space $V$ endowed with a norm $\| \cdot \|$, and we write the pair $(V, \| \cdot \|)$ to indicate it, is called a normed vector space. As mentioned above a typical example is the Euclidean plane.
A normed vector space \((V, \| \cdot \|)\) is a pre-Hilbert space if its norm is defined by
\[
\|v\| = \sqrt{\langle v, v \rangle}, \quad v \in V,
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where \(\langle \cdot, \cdot \rangle : V \times V \to \mathbb{K}\) is a scalar product on \(V\),
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\[\begin{align*}
\langle v, v \rangle &> 0 \text{ for } v \neq 0; \\
\langle v, w \rangle &= \langle w, v \rangle, \text{ for } v, w \in V; \\
\langle u + \lambda v, w \rangle &= \langle u, w \rangle + \lambda \langle v, w \rangle \text{ for } u, v, w \in V \text{ and } \lambda \in \mathbb{K}; \\
\langle w, u + \lambda v \rangle &= \langle w, u \rangle + \overline{\lambda} \langle w, v \rangle \text{ for } u, v, w \in V \text{ and } \lambda \in \mathbb{K}.
\end{align*}\]

The triangle inequality for the norm follows from the Schwarz inequality

\[
|\langle v, w \rangle| \leq \|v\| \|w\|, \quad v, w \in V.
\]
We describe the pre-Hilbert space also by the pair \((V, \langle \cdot, \cdot \rangle)\). A typical example of scalar product over \(\mathbb{K}^I\) is the one we introduced in (1), which generates the Euclidean norm on \(\mathbb{K}^I\):

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\|v\|_2 = \sqrt{\sum_{i \in I} |v_i|^2}, \quad v \in \mathbb{K}^I.
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Scalar products and (pre)-Hilbert spaces

We describe the pre-Hilbert space also by the pair $(V, \langle \cdot, \cdot \rangle)$. A typical example of scalar product over $\mathbb{K}^I$ is the one we introduced in (1), which generates the Euclidean norm on $\mathbb{K}^I$:

$$\| v \|_2 = \sqrt{\sum_{i \in I} |v_i|^2}, \quad v \in \mathbb{K}^I.$$  

As before one can define orthogonality between vectors, orthogonal, and orthonormal sets of vectors.
Hilbert spaces

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Thus, those who are not familiar with topological notions (e.g., completeness, Cauchy sequences, etc.), one should just reason in what follows according to their Euclidean geometric intuition.
Projections

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Given a closed convex set $C$ in a Hilbert space $(V, \langle \cdot, \cdot \rangle)$ one defines the projection of any vector $v$ on $C$ as

$$P_C(v) = \arg \min_{w \in C} \| v - w \|.$$
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This definition is well-posed, as the projection is actually unique, and an equivalent definition is given by fulfilling the following inequality

$$\Re \langle z - P_C(v), v - P_C(v) \rangle \leq 0,$$

for all $z \in C$. This is left as an exercise.
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Moreover, it holds the Pythagoras-Fourier Theorem:
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In case $C = W \subset V$ is actually a closed linear subspace of $V$, then the projection onto $W$ can be readily computed as soon as one disposes of an orthonormal basis for $W$. Let $\{w_\nu\}_{\nu \in \mathcal{F}}$ be a (countable) orthonormal basis for $W$. Let $\{w_\nu\}_{\nu \in \mathcal{F}}$ be a (countable) orthonormal basis for $W$. 
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\|P_W(v)\|^2 = \sum_{\nu \in F} |\langle v, w_\nu \rangle|^2, \text{ for all } v \in V.
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Pythagoras-Fourier Theorem and othonormal expansions

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In this case, obviously, $P_V = I$ and, we have the orthonormal expansion of any vector $v \in \mathcal{V}$,

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and the norm equivalence

$$\|v\|^2 = \sum_{\nu \in F} |\langle v, w_{\nu} \rangle|^2.$$
Trace of a matrix

The effort of defining abstract scalar products and norms allows us now to introduce several norms for matrices.
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First of all we need to introduce the concept of trace, which is the map \( \text{tr} : \mathbb{K}^{l \times l} \to \mathbb{K} \) defined by

\[
\text{tr}(A) = \sum_{i \in I} A_{ii},
\]

i.e., it is the sum of the diagonal elements of the matrix \( A \).
Trace of a matrix: properties

The trace enjoys several properties, which we collect in the following:

1. \( \text{tr}(AB) = \text{tr}(BA) \), for any \( A \in \mathbb{K}^I \times J \) and \( B \in \mathbb{K}^J \times I \);

2. \( \text{tr}(ABC) = \text{tr}(BCA) \), for any \( A, B, C \) matrices of compatible size and indexes; this property is called the circularity property of the trace;

3. As a consequence of the previous property we obtain the invariance of the trace under unitary transformations, i.e., \( \text{tr}(A) = \text{tr}(UAU^H) \) for \( A \in \mathbb{K}^I \times I \) and any unitary matrix \( U \in \mathbb{K}^I \times I \);

4. \( \text{tr}(A) = \sum_{i} \lambda_i \), where \( \lambda_i \) is the set of eigenvalues of \( A \).
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Proposition (Properties of the trace)

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(d) $\text{tr}(A) = \sum_{i \in I} \lambda_i$, where $\{\lambda_i : i \in I\}$ is the set of eigenvalues of $A$. 
Matrix norms

The Frobenius norm of a matrix is essentially the Euclidean norm computed over the entries of the matrix (considered as a vector):

\[ A_F = \sqrt{\sum_{i} \sum_{j} |A_{ij}|^2}, \]

It is also known as the Schur norm or the Hilbert-Schmidt norm. This norm is generated by the scalar product (that's why we made the effort of introducing abstract scalar products!):

\[ x^T A^T B y_F = \text{tr}(AB^H) = \text{tr}(B^HA^H), \]

In particular,

\[ A_F^2 = \text{tr}(A^H A), \]

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$$\|A\|_F := \sqrt{\sum_{i \in I, j \in J} |A_{ij}|^2}, \quad A \in \mathbb{K}^{I \times J}.$$ 

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$$\langle A, B \rangle_F := \sum_{i \in I} \sum_{j \in J} A_{ij} \overline{B_{ij}} = \text{tr}(AB^H) = \text{tr}(B^H A). \quad (4)$$
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Let $\| \cdot \|_X$ and $\| \cdot \|_Y$ be vector norms on the vector spaces $X = \mathbb{K}^I$ and $Y = \mathbb{K}^J$, respectively. If both $\| \cdot \|_X$ and $\| \cdot \|_Y$ coincides with the Euclidean norms $\| \cdot \|_2$ on $X = \mathbb{K}^I$ and $Y = \mathbb{K}^J$, respectively, then the associated matrix norm is called the spectral norm and it is denoted with $\| \cdot \|_2$.
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$$\| A \| := \| A \|_{X \rightarrow Y} := \sup_{z \neq 0} \frac{\| A z \|_Y}{\| z \|_X}, \quad A \in \mathbb{K}^{I \times J}.$$
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As the spectral norm has a central importance in many applications, it is often denoted just by $\| A \|$.
Unitary invariance and submultiplicativity

Both the matrix norms we introduced so far are invariant with respect to unitary transformations, i.e., $\|A\|_F = \|UAV^H\|_F$ and $\|A\|_\infty = \|UAV^H\|_\infty$ for unitary matrices $U, V$. 
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Moreover, both are submultiplicative, i.e.,

$\|AB\|_F \leq \|A\|_\infty \|B\|_F \leq \|A\|_F \|B\|_F$ and $\|AB\| \leq \|A\| \|B\|$, for matrices $A, B$ of compatible sizes.
We introduce and analyze the singular value decomposition (SVD) of a matrix $A$, which is the factorization of $A$ into the product of three matrices $A = U\Sigma V^H$, where $U$, $V$ are orthogonal matrices of compatible size and the matrix $\Sigma$ is diagonal with positive real entries.
Introduction of the Singular Value Decomposition

We introduce and analyze the singular value decomposition (SVD) of a matrix $A$, which is the factorization of $A$ into the product of three matrices $A = U\Sigma V^H$, where $U$, $V$ are orthogonal matrices of compatible size and the matrix $\Sigma$ is diagonal with positive real entries.
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All these terms, orthogonal, diagonal matrix, have been introduced in the previous lectures.
To gain insight into the SVD, treat the $n = \#l$ rows of a matrix $A \in \mathbb{K}^{l \times J}$ as points in a $d$-dimensional space, where $d = \#J$, and consider the problem of finding the best $k$-dimensional subspace with respect to the set of points.
To gain insight into the SVD, treat the $n = \#_I$ rows of a matrix $A \in \mathbb{K}^{I \times J}$ as points in a $d$-dimensional space, where $d = \#_J$, and consider the problem of finding the best $k$-dimensional subspace with respect to the set of points. Here best means minimize the sum of the squares of the perpendicular distances of the points to the subspace. An orthonormal basis for this subspace is built as fundamental directions with maximal variance of the group of high dimensional points, and are called principal components.
Geometrical derivation: principal component analysis
We begin with a special case of the problem where the subspace is 1-dimensional, a line through the origin.
Pythagoras Theorem, scalar products, and orthogonal projections

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We will see later that the best-fitting $k$-dimensional subspace can be found by $k$ applications of the best fitting line algorithm.
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We will see later that the best-fitting $k$-dimensional subspace can be found by $k$ applications of the best fitting line algorithm.

Finding the best fitting line through the origin with respect to a set of points $\{x_i := A(i) \in \mathbb{K}^2 : i \in I\}$ in the Euclidean plane $\mathbb{K}^2$ means minimizing the sum of the squared distances of the points to the line.
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Pythagoras Theorem, scalar products, and orthogonal projections

In the best least squares fit, one is minimizing the distance to a subspace. Now, consider projecting orthogonally a point \( x_i \) onto a line through the origin. Then, by Pythagoras theorem

\[
x_{i1}^2 + x_{i2}^2 + \cdots + x_{id}^2 = \text{(length of projection)}^2 + \text{(distance of pt. to line)}^2.
\]
In particular, from the formula above, one has

\[(\text{distance of pt to line})^2 = x_{i1}^2 + x_{i2}^2 + \cdots + x_{id}^2 - (\text{length of projection})^2.\]
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To minimize the sum of the squares of the distances to the line, one could minimize \(\sum_i (x_{i1}^2 + x_{i2}^2 + \cdots + x_{id}^2)\) minus the sum of the squares of the lengths of the projections of the points to the line.
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However, the first term of this difference is a constant (independent of the line), so minimizing the sum of the squares of the distances is equivalent to maximizing the sum of the squares of the lengths of the projections onto the line.
Pythagoras Theorem, scalar products, and orthogonal projections
Averaging through the points and matrix notation

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Let $v$ be a unit vector along this line. The length of the projection of $A^{(i)}$, the $i^{th}$ row of $A$, onto $v$ is, according to our definition of scalar product, $|\langle A^{(i)}, v \rangle|$. 
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Let $\nu$ be a unit vector along this line. The length of the projection of $A^{(i)}$, the $i^{th}$ row of $A$, onto $\nu$ is, according to our definition of scalar product, $|\langle A^{(i)}, \nu \rangle|$.

From this, we see that the sum of length squared of the projections is

$$\|Av\|_2^2 = \sum_{i \in I} |\langle A^{(i)}, \nu \rangle|^2.$$  \hspace{1cm} (6)
Best-fit and first singular direction

The best-fit line is the one maximizing $\|Av\|_2^2$ and hence minimizing the sum of the squared distances of the points to the line.
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With this in mind, define the first singular vector, \( v_1 \) of \( A \), which is a column vector, as the best-fit line through the origin for the \( n \) points in \( d \)-dimensional space that are the rows of \( A \).
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Thus

$$v_1 = \arg \max_{\|v\|_2=1} \|Av\|_2.$$
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$$v_1 = \arg \max_{\|v\|_2=1} \|Av\|_2.$$

The value $\sigma_1(A) = \|Av_1\|_2$ is called the first singular value of $A$. Note that $\sigma_1(A)^2$ is the sum of the squares of the projections of the points to the line determined by $v_1$. 
This is the first time we use an optimization criterion to approach a data interpretation problem and to create a connection between linear algebra and optimization.
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As we will discuss along this course, optimization plays indeed a huge role in data analysis and we will have to discuss it in more detail later on.
A greedy approach to best $k$-dimensional fit

The greedy approach to find the best-fit 2-dimensional subspace for a matrix $A$, takes $v_1$ as the first basis vector for the 2-dimensional subspace and finds the best 2-dimensional subspace containing $v_1$. The fact that we are using the sum of squared distances will again help, thanks to Pythagoras Theorem: For every 2-dimensional subspace containing $v_1$, the sum of squared lengths of the projections onto the subspace equals the sum of squared projections onto $v_1$ plus the sum of squared projections along a vector perpendicular to $v_1$ in the subspace. Thus, instead of looking for the best 2-dimensional subspace containing $v_1$, look for a unit vector, call it $v_2$ that maximizes $\|Av\|_2^2$ among all such unit vectors. Using the same greedy strategy to find the best three and higher dimensional subspaces, defines $v_3$, $v_4$, ... in a similar manner.
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Using the same greedy strategy to find the best three and higher dimensional subspaces, defines $v_3, v_4, \ldots$ in a similar manner.
Formal definitions

More formally, the second singular vector, $v_2$ is defined by the best-fit line perpendicular to $v_1$:

$$v_2 = \arg \max_{\|v\|=1, \langle v_1, v \rangle = 0} \|Av\|_2.$$
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More formally, the second singular vector, $\mathbf{v}_2$ is defined by the best-fit line perpendicular to $\mathbf{v}_1$:

$$\mathbf{v}_2 = \arg \max_{\|\mathbf{v}\|=1, \langle \mathbf{v}_1, \mathbf{v} \rangle = 0} \| \mathbf{A} \mathbf{v} \|_2.$$ 

The value $\sigma_2(\mathbf{A}) = \| \mathbf{A} \mathbf{v}_2 \|_2$ is called the second singular value of $\mathbf{A}$. 

and so on.

The process stops when we have found $\mathbf{v}_1, \ldots, \mathbf{v}_r$ as singular vectors and $\sigma_r(\mathbf{A}) = \| \mathbf{A} \mathbf{v}_r \|_2$. 

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The value $\sigma_2(A) = \|Av_2\|_2$ is called the second singular value of $A$. The $k^{th}$ singular vector $v_k$ is defined similarly by

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Optimality of the greedy algorithm

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Optimality of the greedy algorithm

If instead of finding \( v_1 \) that maximized \( \|Av\|_2 \) and then the best-fit 2-dimensional subspace containing \( v_1 \), we had found the best-fit 2-dimensional subspace, we might have done better.

Surprisingly enough, this is actually not the case. We now give a simple proof that the greedy algorithm indeed finds the best subspaces of every dimension.
If instead of finding $v_1$ that maximized $\|Av\|_2$ and then the best-fit 2-dimensional subspace containing $v_1$, we had found the best-fit 2-dimensional subspace, we might have done better.

Surprisingly enough, this is actually not the case. We now give a simple proof that the greedy algorithm indeed finds the best subspaces of every dimension.

**Proposition**

Let $A \in \mathbb{K}^{I \times J}$ and $v_1, v_2, \ldots, v_r$ be the singular vectors defined above. For $1 \leq k \leq r$, let $V_k$ be the subspace spanned by $v_1, v_2, \ldots, v_k$. Then for each $k$, $V_k$ is the best-fit $k$-dimensional subspace for $A$. 
We conclude this part with an important property left as an exercise: show that necessarily

\[ \sigma_1(A) \geq \sigma_2(A) \geq \cdots \geq \sigma_r(A), \]  

i.e., the singular values come always with a natural nonincreasing order.
On matrix norms again

Note that the vector $Av_i$ is really a list of lengths (with signs) of the projections of the rows of $A$ onto $v_i$. 
Note that the vector $A\nu_i$ is really a list of lengths (with signs) of the projections of the rows of $A$ onto $\nu_i$. Think of $\sigma_i(A) = \|A\nu_i\|_2$ as the component of the matrix $A$ along $\nu_i$. 
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For this interpretation to make sense, it should be true that adding up the squares of the components of $A$ along each of the $v_i$ gives the square of the whole content of the matrix $A$. 
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For this interpretation to make sense, it should be true that adding up the squares of the components of $A$ along each of the $\nu_i$ gives the square of the whole content of the matrix $A$.

This is indeed the case and is the matrix analogy of decomposing a vector into its components along orthogonal directions.
On matrix norms again

Consider one row, say $A^{(i)}$ of $A$. 
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\[ \|A^{(i)}\|_2 \leq \sqrt{\sum_{j=1}^{r} |A_{ij}|^2} \] 

Summing over all rows $i \in I$ and recalling (6), we obtain 

\[ \|A\|_2 \leq \sqrt{\sum_{i=1}^{r} \|A^{(i)}\|_2^2} \] 

But 

\[ \|A\|_2 \leq \sqrt{\sum_{i=1}^{r} \|A^{(i)}\|_2^2} \] 

is the Frobenius norm of $A$, so that we obtained 

\[ \|A\|_F \leq \sqrt{\sum_{k=1}^{r} \sigma_k^2} \] 

(9)
On matrix norms again

Consider one row, say $A^{(i)}$ of $A$. Since $v_1, v_2, \ldots, v_r$ span the space of all rows of $A$ (exercise!), $\langle A^{(i)}, v \rangle = 0$ for all $v$ orthogonal to $v_1, v_2, \ldots, v_r$. 
On matrix norms again

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Thus, for each row $A^{(i)}$, by Pythagoras Theorem,

$$\|A^{(i)}\|_2^2 = \sum_{k=1}^{r} |\langle A^{(i)}, v_k \rangle|^2.$$
On matrix norms again

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On matrix norms again

Consider one row, say $A(i)$ of $A$. Since $v_1, v_2, \ldots, v_r$ span the space of all rows of $A$ (exercise!), $\langle A(i), v \rangle = 0$ for all $v$ orthogonal to $v_1, v_2, \ldots, v_r$.

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\]

But
\[
\sum_{i \in I} \|A(i)\|_2^2 = \sum_{i \in I} \sum_{j \in J} |A_{ij}|^2,
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is the Frobenius norm of $A$,
On matrix norms again

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\|A\|_F = \sqrt{\sum_{k=1}^r \sigma_k(A)^2}.
\] (9)
On matrix norms again

We might also observe that

$$\sigma_1(A) = \max_{\|v\|_2 = 1} \|Av\|_2 = \arg \max_{v \neq 0} \frac{\|Av\|_2}{\|v\|_2} = \|A\|,$$  \hspace{1cm} (10)

i.e., the first singular vector corresponds to the spectral norm of $A$. 

Notice that these definitions give precisely

$$\|A\|_F$$

and

$$\|A\|_2$$

as particular Schatten-norms.

Of particular relevance in certain applications related to recommender systems is the so-called nuclear norm

$$\|A\|_1$$

corresponding to the Schatten-1-norm.
On matrix norms again

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Notice that these definitions gives precisely $\|A\|_F = \|A\|_2$ and $\|A\| = \|A\|_\infty$ as particular Schatten-norms. Of particular relevance in certain applications related to recommender systems is the so-called nuclear norm $\|A\|_* = \|A\|_1$ corresponding to the Schatten-1-norm.
On rank again

We just mentioned above, and left it as an exercise, that the orthogonal basis constituted by the vectors $v_1, v_2, \ldots, v_r$ spans the space of all rows of $A$. 
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This means that the number \( r \) (of them) is actually the rank of the matrix! And this a proof of (g) in the Proposition above characterizing the rank!
The Singular Value Decomposition in its full glory

The vectors $Av_1, Av_2, \ldots, Av_r$ form yet another fundamental set of vectors associated with $A$: 

$$u_i = \frac{Av_i}{\sigma_i}$$
The Singular Value Decomposition in its full glory

The vectors $A v_1, A v_2, \ldots, A v_r$ form yet another fundamental set of vectors associated with $A$: We normalize them to length one by setting

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The SVD theorem (soon!) will fully explain the reason for these terms.
Orthonormality of the left singular vectors

Clearly, the right singular vectors are orthogonal by definition.
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We now show that the left singular vectors are also orthogonal.

**Theorem**

Let \( A \in \mathbb{K}^{I \times J} \) be a rank \( r \) matrix. The left singular vectors of \( A \), \( u_1, u_2, \ldots, u_r \) are orthogonal.
SVD in its full glory

**Theorem**

Let $A \in \mathbb{K}^{I \times J}$ with right singular vectors $v_1, v_2, \ldots, v_r$, left singular vectors $u_1, u_2, \ldots, u_r$, and corresponding singular values $\sigma_1, \ldots, \sigma_r$. Then

$$A = \sum_{k=1}^{r} \sigma_k u_k v_k^H$$
Best rank-\(k\) approximation

Given the singular value decomposition \(A = \sum_{k=1}^{r} \sigma_k u_k v_k^H\) of a matrix \(A \in \mathbb{K}^{I \times J}\),
Best rank-$k$ approximation

Given the singular value decomposition $A = \sum_{k=1}^{r} \sigma_k u_k v_k^H$ of a matrix $A \in \mathbb{K}^{l \times j}$, we define the $k$-rank truncation by

$$A_k = \sum_{\ell=1}^{k} \sigma_{\ell} u_{\ell} v_{\ell}^H.$$
Best rank-$k$ approximation

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It should be clear that $A_k$ is a $k$-rank matrix.
Best rank-\( k \) approximation

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It should be clear that \( A_k \) is a \( k \)-rank matrix.

Lemma

The rows of \( A_k \) are the orthogonal projections of the rows of \( A \) onto the subspace \( V_k \) spanned by the first \( k \) singular vectors of \( A \).
Best rank-\(k\) Frobenius approximation

**Theorem**

*For any matrix \(B\) of rank at most \(k\)*

\[
\| A - A_k \|_F \leq \| A - B \|_F
\]
Best rank-$k$ spectral approximation

Lemma

$\|A - A_k\|^2 = \sigma_{k+1}^2$. 
Best rank-$k$ spectral approximation

**Lemma**
\[ \|A - A_k\|^2 = \sigma_{k+1}^2. \]

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*For any matrix $B$ of rank at most $k$*
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We introduced the SVD by means of a greedy algorithm, which at each step is supposed to solve a nonconvex optimization problem of the type

\[ v_k = \arg \max_{\|v\|_2 = 1, \langle v_1, v \rangle = 0, \ldots, \langle v_{k-1}, v \rangle = 0} \| Av \|_2. \]  

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Nonconvexity and curse of dimensionality

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How can we then approach the computation of the SVD without incurring in the so called “curse of dimensionality”, which is an emphatic way of describing the computational infeasibility of a certain numerical task?
The Power Method

It is easiest to describe first in the case when $A$ is real-valued, square, and symmetric and has the same right and left singular vectors, namely, $A = \sum_{k=1}^{r} \sigma_k \nu_k \nu_k^T$. 
The Power Method

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In this case we have

$$A^2 = \left( \sum_{k=1}^{r} \sigma_k v_k v_k^T \right) \left( \sum_{\ell=1}^{r} \sigma_\ell v_\ell v_\ell^T \right) = \sum_{k, \ell=1}^{r} \sigma_\ell \sigma_k v_k \underbrace{v_\ell^T v_\ell}_{:= \delta_{k\ell}} v_\ell^T = \sum_{k=1}^{r} \sigma_k^2 v_k v_k^T.$$
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Similarly, if we take the $m^{th}$ power of $A$, again all the cross terms are zero and we will get

$$A^m = \sum_{k=1}^{r} \sigma_k^m v_k v_k^T.$$
We had the spectral norm of $A$ and a spectral gap

If we had $\sigma_1 \gg \sigma_2$, we would have

$$\lim_{m \to \infty} \frac{A^m}{\sigma_1^m} = \nu_1 \nu_1^T.$$
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While it is not so simple to compute $\sigma_1$ (why?!), which corresponds to the spectral norm of $A$, one can easily compute the Frobenius norm $\|A^m\|_F$, so that we can consider $\frac{A^m}{\|A^m\|_F}$ which again converges for $m \to \infty$ to $v_1 v_1^T$ from which $v_1$ may be computed (exercise!).
What if $A$ is not squared?

But then $B = AA^T$ is squared.
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This is the spectral decomposition of $B$. Using the same kind of calculation as above, one obtains

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As $m$ increases, for $k > 1$ the ratio $\sigma_k^{2m}/\sigma_1^{2m}$ goes to zero and $B^m$ gets approximately equal to

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provided again that $\sigma_1 \gg \sigma_2$. This suggests how to compute both $\sigma_1$ and $u_1$, simply by powering $B$. 
Two relevant issues

- If we do not have a spectral gap $\sigma_1 \gg \sigma_2$ we get into troubles.
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- If we do not have a spectral gap $\sigma_1 \gg \sigma_2$ we get into troubles.
- Computing $B^m$ costs $m$ matrix-matrix multiplication, when done in the schoolbook way it costs $O(md^3)$ operations or $O(m \log d)$ when done by successive squaring.
Use a **random** vector application instead!

Instead, we compute

\[ B^m x, \]

where \( x \) is a random unit length vector.
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The idea is that the component of \( x \) in the direction of \( u_1 \), i.e.,
\[ x_1 := \langle x, u_1 \rangle \]
is actually bounded away from zero with high probability and would get multiplied by \( \sigma_1^2 \), while the other components of \( x \) along other \( u_i \)'s would be multiplied by \( \sigma_k^2 \ll \sigma_1^2 \) only.
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Each increase in \( m \) requires multiplying \( B \) by the vector \( B^{m-1} x \),
which we can further break up into

\[ B^m x = A(A^T(B^{m-1} x)). \]

This requires two matrix vector products, involving the matrices \( A^T \) and \( A \).
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This requires two matrix vector products, involving the matrices \( A^T \) and \( A \). Since \( B^m x \approx \sigma_1^{2k} u_1(u_1^T x) \) is a scalar multiple of \( u_1 \), \( u_1 \) can be recovered from \( B^m x \) by normalization.
The concentration of measure phenomenon informally speaking is describing the fact that - in high-dimension - certain events happen with high-probability.
Concentration of measure phenomenon

The concentration of measure phenomenon informally speaking is describing the fact that - in high-dimension - certain events happen with high-probability. In particular, we have

**Lemma**

Let \( x \in \mathbb{R}^d \) be a unit d-dimensional vector of components \( x = (x_1, \ldots, x_d) \) with respect to the canonical basis and picked at random from the set \( \{ x : \| x \|_2 \leq 1 \} \). The probability that \( |x_1| \geq \alpha > 0 \) is at least \( 1 - 2\alpha \sqrt{d - 1} \).
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Remark

Notice that in the previous result essentially shows also that, independently of the dimension $d$, the $x_1 = \langle x, u_1 \rangle$ component of a random unit vector $x$ with respect to any orthonormal basis $\{u_1, \ldots, u_d\}$ is bounded away from zero with overwhelming probability.

\[ \text{Since the sphere is rotation invariant, the arguments above apply to any orthonormal basis by rotating it to coincide with the canonical basis!} \]
Extensions

Remark
Notice that in the previous result essentially shows also that, independently of the dimension $d$, the $x_1 = \langle x, u_1 \rangle$ component of a random unit vector $x$ with respect to any orthonormal basis $\{u_1, \ldots, u_d\}$ is bounded away from zero with overwhelming probability.

Remark
In view of the isometrical mapping $(a, b) \rightarrow a + ib$ from $\mathbb{R}^2$ to $\mathbb{C}$ the previous result extends to random unit vectors in $\mathbb{C}^d$ simply by modifying the statement as follows: The probability that, for a randomly chosen unit vector $z \in \mathbb{C}^d$ $|z_1| \geq \alpha > 0$ holds is at least $1 - 2\alpha \sqrt{2d - 1}$.

\footnote{Since the sphere is rotation invariant, the arguments above apply to any orthonormal basis by rotating it to coincide with the canonical basis!}
Randomized Power Method

By injecting randomization in the process, we are able to use the blessing of the dimensionality (concentration of measure) against the curse of dimensionality.
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**Theorem**

Let $A \in \mathbb{K}^{I \times J}$ and $x \in \mathbb{K}^I$ be a random unit length vector. Let $V$ be the space spanned by the left singular vectors of $A$ corresponding to singular values greater than $(1 - \varepsilon)\sigma_1$. Let $m$ be $\Omega \left( \frac{\ln(d/\varepsilon)}{\varepsilon} \right)$. Let $w^*$ be unit vector after $m$ iterations of the power method, namely,

$$w^* = \frac{(AA^H)^m x}{\| (AA^H)^m x \|_2}.$$

The probability that $w^*$ has a component of at least $\mathcal{O} \left( \frac{\varepsilon}{\alpha d} \right)$ orthogonal to $V$ is at most $1 - 2\alpha\sqrt{2d - 1}$. 
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To make the previous statement concrete, let's take \( \alpha = \frac{1/10}{2d} \).

Then, with probability at least 9/10 (i.e., very close to 1), after \( m \) iteration of order \( \Omega\left(\frac{\ln(d/\varepsilon)}{\varepsilon}\right) \) the vector \( w^* \) has a component of at least \( \mathcal{O}(\varepsilon) \) orthogonal to \( V \).
Applications of the SVD: Principal Component Analysis

The traditional use of SVD is in Principal Component Analysis (PCA).
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Let matrix $A$ with elements $A_{ij}$ represent the probability of customer $i$ purchasing product $j$ (or the amount or utility of product $j$ to customer $i$).
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An individual customer’s behavior is determined by some weighted combination of these underlying factors.
Principal Component Analysis

That is, a $i$ customer’s purchase behavior can be characterized by a $k$-dimensional vector $(u_{\ell,i})_{\ell=1,...,k}$, where $k$ is much smaller than $n$ and $d$. 
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Associated with each basic factor $\ell$ is a vector of probabilities $v_{\ell}$, each component of which is the probability of purchasing a given product by someone whose behavior depends only on that factor.
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More abstractly, $A = UV^H$ is an $n \times d$ matrix that can be expressed as the product of two matrices $U$ and $V$ where $U$ is an $n \times k$ matrix expressing the factor weights for each customer and $V$ is a $k \times d$ matrix expressing the purchase probabilities of products that correspond to that factor:
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\[
(A^{(i)})^H = \sum_{\ell=1}^{k} u_{\ell,i} v_\ell^H.
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One twist is that $A$ may not be exactly equal to $UV^T$, but close to it since there may be noise or random perturbations.
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Taking the best rank-$k$ approximation $A_k$ from SVD (as described above) gives us such a $U, V$. In this traditional setting, one assumed that $A$ was available fully and we wished to find $U, V$ to identify the basic factors or in some applications to denoise $A$ (if we think of $A - UV^T$ as noise).
Low rankness and recommender systems

Now imagine that $n$ and $d$ are very large, on the order of thousands or even millions, there is probably little one could do to estimate or even store $A$. If $A$ was an arbitrary matrix of size $n \times d$, this would require $\Omega(pnd)$ pieces of information and cannot be done with a few entries. But again hypothesize that $A$ was a small rank matrix with added noise. If now we also assume that the given entries are randomly drawn according to some known distribution, then there is a possibility that SVD can be used to estimate the whole of $A$. This area is called collaborative filtering or low-rank matrix completion, and one of its uses is to target an ad to a customer based on one or two purchases.
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SVD as a compression algorithm

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Thus, one could use SVD as a compression method.
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Sparsifying bases

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We will discuss later on the use of sparsifying bases.
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A collection of $n$ documents may be represented by a collection of 25,000-dimensional vectors, one vector per document. The vectors may be arranged as rows $n \times 25,000$ matrix.
Singular Vectors and ranking documents

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Is there a better weighting of terms, i.e., a better projection direction which would measure the intrinsic relevance of the document to the collection?
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An intuitive reason for this is that this direction has the maximum sum of squared projections of the collection and so can be thought of as a synthetic term-document vector best representing the document collection.
WWW ranking

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Some web pages, called authorities, are the most prominent sources for information on a given topic. Other pages called hubs, are ones that identify the authorities on a topic.
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If there are $n$ nodes, the hub weights form a $n$-dimensional vector $u$ and the authority weights form a $n$-dimensional vector $v$. Suppose $A$ is the adjacency matrix representing the directed graph: $A_{ij}$ is 1 if there is a hypertext link from page $i$ to page $j$ and 0 otherwise. Given hub vector $u$, the authority vector $v$ could be computed by the formula

$$v_j = \sum_{i=1}^{n} u_i a_{ij},$$

since the right hand side is the sum of the hub weights of all the nodes that point to node $j$. 
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since the right hand side is the sum of the hub weights of all the nodes that point to node $j$. In matrix terms,

$$v = A^T u.$$

Similarly, given an authority vector $v$, the hub vector could be computed by $u = Av$. 
Of course, at the start, we have neither vector.
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But the above suggests a power-like iteration: Start with any $v$. Set $u = Av$; then set $v = A^T u$, and repeat the process.
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We know from the power method that this converges to the left and right singular vectors. So, after sufficiently many iterations, we may use the left vector \( u \) as hub weights vector and project each column of \( A \) onto this direction and rank columns (authorities) in order of this projection.
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But the projections just form the vector \( A^T u \) which equals \( v \). So we can just rank by order of \( v_j \).

This is the basis of an algorithm called the HITS algorithm which was one of the early proposals for ranking web pages. A different ranking called page rank is widely used. It is based on a random walk on the graph described above, but we do not explore it here.
Pseudo-inverse matrices

Given \( A \in \mathbb{K}^{I \times J} \) of rank \( r \) and its singular value decomposition

\[
A = U \Sigma V^H = \sum_{k=1}^{r} \sigma_k u_k v_k^H,
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we define the Moore-Penrose pseudo-inverse \( A^+ \) as

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A^+ = V \Sigma^+ U^H,
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where the inverse \( \Sigma^+ \) is applied only to the nonzero singular values of \( A \).
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In this case, it should be also noted two fundamental properties of $A^\dagger$. First of all, it is a left inverse of $A$, i.e.,

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where $I \in \mathbb{K}^{r \times r}$. On the other side the matrix

$$P_{\text{range}(A)} = AA^\dagger,$$

corresponds to the orthogonal projection of $\mathbb{K}^I$ onto the range of the matrix $A$, i.e., the span on the columns of $A$. 
Similarly, if $AA^H \in \mathbb{K}^{I \times I}$ is invertible (implying $\#I \leq \#J$) and this is true if $r = \#I$ and the rows are linearly independent,
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In this case, $A^\dagger$ is a right inverse of $A$, i.e.,

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Least square problems

- The system $Ax = y$ is overdetermined, i.e., there are more equations than unknowns. In this case the problem might even not be solvable and approximated solutions are needed, i.e., those that minimize the mismatch $\|Ax - y\|_2$;
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- The system $Ax = y$ is underdetermined, i.e., there are less equations than unknowns. In this case the problem has an infinite number of solutions and one would wish to determine one of them with special properties, by means of a proper selection criterion.
Least square problems

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- The system $Ax = y$ is underdetermined, i.e., there are less equations than unknowns. In this case the problem has an infinite number of solutions and one would wish to determine one of them with special properties, by means of a proper selection criterion. A criterion of selection of proper solutions of the systems $Ax = y$ is often to pick the solution with minimal Euclidean norm $\|x\|_2$. 
Pseudo inverse matrices and Least square problems

Let us first connect least squares problems with the Moore-Penrose pseudo-inverse introduced above.

**Proposition**

Let $A \in \mathbb{K}^{I \times J}$ and $y \in \mathbb{K}^I$. Define $\mathcal{M} \subset \mathbb{K}^J$ to be the set of minimizers of the map $x \to \|Ax - y\|_2$. The convex optimization problem

$$\arg \min_{x \in \mathcal{M}} \|x\|_2 \tag{12}$$

has the unique solution $x^* = A^\dagger y$. 

Proof at the blackboard...
Pseudo inverse matrices and Least square problems

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Proof at the blackboard ...
Overdetermined systems

Corollary

Let $A \in \mathbb{K}^{I \times J}$ with $n = \#I \geq \#J = d$ be of full rank $r = d$, and let $y \in \mathbb{K}^I$. The least square problem

$$\arg \min_{x \in \mathbb{K}^J} \|Ax - y\|_2,$$

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Corollary

Let $A \in \mathbb{K}^{I \times J}$ with $n = \# I \geq \# J = d$ be of full rank $r = d$, and let $y \in \mathbb{K}^{I}$. The least square problem

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has the unique solution $x = A^\dagger y$.

This result follows from the Proposition above because the minimizer $x^*$ of $\|Ax - y\|_2$ is in this case unique.
Corollary

Let $A \in \mathbb{K}^{I \times J}$ with $n = \#I \leq \#J = d$ be of full rank $r = n$, and let $y \in \mathbb{K}^I$. The least square problem

$$\arg \min_{x \in \mathbb{K}^J} \|x\|_2 \text{ subject to } Ax = y,$$

has the unique solution $x = A^\dagger y$.  

(14)
Stability of the Singular Value Decomposition

In many situations, there is a data matrix of interest $A$, but one has only a perturbed version of it $\tilde{A} = A + E$ where $E$ is going to be a relatively small quantifiable perturbation.
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What happens if we acquire more and more data and the matrices become larger and larger?

Also to be able to respond in a quantitative way to such questions we need to develop a theory of stability of singular value decompositions under small perturbations.
Spectral theorem

A ∈ \( \mathbb{K}^{I \times I} \) is actually a Hermitian matrix, i.e. \( A = A^H \).
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$A \in \mathbb{K}^{l \times l}$ is actually a Hermitian matrix, i.e. $A = A^H$. We recall that a nonzero vector $v \in \mathbb{K}^l$ is an eigenvector of $A$ if $Av = \lambda v$ for some scalar $\lambda \in \mathbb{K}$ called the corresponding eigenvalue.
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**Theorem (Spectral theorem for Hermitian matrices)**

*If $A \in \mathbb{K}^{I \times I}$ and $A = A^H$, then there exists an orthonormal basis $\{v_1, \ldots v_n\}$ consisting of eigenvectors of $A$ with real corresponding eigenvalues $\{\lambda_1, \ldots \lambda_n\}$ such that

\[
A = \sum_{k=1}^{n} \lambda_k v_k v_k^H.
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*The previous representation is called the spectral decomposition of $A$.*
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In fact, by denoting $u_k = \text{sign} \lambda_k$ and $\sigma_k = |\lambda_k|$, then

$$A = \sum_{k=1}^{n} \lambda_k v_k v_k^H = \sum_{k=1}^{n} \text{sign} \lambda_k |\lambda_k| v_k v_k^H = \sum_{k=1}^{n} \sigma_k u_k v_k^H,$$

which is actually the SVD of $A$. 
Weyl’s bounds

Assume now $E$ is Hermitian so that $\tilde{A} = \tilde{A}^H$ holds, as well.
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$$\lambda_1(\tilde{A}) = \max_{\|v\|_2=1} v^H (A + E) v$$

$$\leq \max_{\|v\|_2=1} v^H A v + \max_{\|v\|_2=1} v^H E v$$

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By using the spectral decomposition of $A + E$ do prove as an exercise the first equality above.
Weyl’s bounds

Let $\nu_1$ be the eigenvector of $A$ associated to the top eigenvalue of $A$. 
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\geq v_1^H(A + E)v_1
= \lambda_1(A) + v_1^HEv_1
\geq \lambda_1(A) + \lambda_n(E).
$$

From these estimates we obtain the bounds

$$
\lambda_1(A) + \lambda_n(E) \leq \lambda_1(A + E) \leq \lambda_1(A) + \lambda_1(E).
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Weyl’s bounds

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Theorem (Weyl)

If $A, E \in \mathbb{K}^{I \times I}$ are two Hermitian matrices, then for all $k = 1, \ldots, n$

$$\lambda_k(A) + \lambda_n(E) \leq \lambda_k(A + E) \leq \lambda_k(A) + \lambda_1(E).$$
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This is a very strong result because it does not depend at all from the magnitude of the perturbation matrix $E$. As a corollary

**Corollary**

*If* $A, E \in \mathbb{K}^{l \times l}$ *are two arbitrary matrices, then for all* $k = 1, \ldots, n$

$$|\sigma_k(A + E) - \sigma_k(A)| \leq \|E\|,$$

*where* $\sigma_k$’s *are the corresponding singular vectors and* $\|E\|$ *is the spectral norm of* $E$.  

Mirsky’s bounds

Actually more can be said:

**Theorem (Mirsky)**

*If* $A, E \in \mathbb{K}^{l \times l}$ *are two arbitrary matrices, then*

$$\sqrt{\sum_{k=1}^{n} |\sigma_k(A + E) - \sigma_k(A)|^2} \leq \|E\|_F,$$

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*where $\|E\|_F$ is the Frobenius norm of $E$.*

In its original form, Mirsky’s theorem holds for an arbitrary unitarily invariant norm (hence for the spectral norm) and includes Weyl’s theorem as a special case.
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$$\cos \theta(v, w) = \frac{w^T v}{\|v\|_2 \|w\|_2}.$$
Stability of singular spaces: principal angles

To quantify the stability properties of the singular value decomposition under perturbations, we need to introduce the notion of principal angles between subspaces. To introduce the concept we recall that in a Euclidean space $\mathbb{R}^d$ the angle $\theta(v, w)$ of two vectors $v, w \in \mathbb{R}^d$ is defined by

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In particular, for unitary vectors $v, w$ one has $\cos \theta(v, w) = w^T v$. 

Stability of singular spaces: principal angles

Definition
Let $V, W \in \mathbb{K}^{J \times I}$ be orthogonal matrices spanning $V$ and $W$, $n$-dimensional subspaces of $\mathbb{K}^J$ (here we may assume that $n = \# I \leq \# J = d$). We define the principal angles between $V$ and $W$ by

$$\cos \Theta(V, W) = \Sigma(W^H V),$$

where $\Sigma(W^H V)$ is the list of the singular values of the matrix $W^H V$.

---

$^3$Here we committed an abuse of notation as we denoted with $V, W$ the matrices of the orthonormal bases of two subspaces $V, W$. However, this identification here is legitimated as all we need to know about the subspaces are some orthonormal bases of them.
Stability of singular spaces: principal angles

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\[ P_V = VV^H, \quad P_W = WW^H. \]
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The following result essentially says that the principal angles quantify how different are the orthogonal projections onto \( V \) and \( W \).

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**Proposition**

*Let $V, W \in \mathbb{K}^{J \times I}$ be orthogonal matrices spanning $V$ and $W$, $n$-dimensional subspaces of $\mathbb{K}^J$ (here we may assume that $n = \# I \leq \# J = d$). Then*

$$ \| P_V - P_W \|_F = \| VV^H - WW^H \|_F = \sqrt{2} \| \sin \Theta(V, W) \|_2. $$

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Stability of singular spaces: principal angles

The orthogonal space $W_{\perp}$ to any subspace $W$ of $\mathbb{K}^J$ is defined by $W_{\perp} = \{ v \in \mathbb{K}^J : \langle v, w \rangle = 0, \ \forall w \in W \}$. 
Stability of singular spaces: principal angles

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$$P_W = I - P_{W_\perp} \quad \text{or} \quad WW^H = I - W_\perp W_{\perp}^H. \quad (15)$$
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Let $V, W \in \mathbb{K}^{J \times I}$ be orthogonal matrices spanning $V$ and $W$, $n$-dimensional subspaces of $\mathbb{K}^J$ (here we may assume that $n = \#I \leq \#J = d$). We denote $W_{\perp}$ an orthogonal matrices spanning $W_{\perp}$. Then

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$$\|W_\perp^H V\|_F = \|\sin \Theta(V, W)\|_2.$$ 

Proof at the blackboard ...
We leave as an additional exercise to show that, as a corollary of the previous result, one has also

$$\| P_{W \perp} P_V \|_F = \| \sin \Theta(V, W) \|_2. \quad (16)$$

Again, one has to use the properties of the trace operator!
Stability of singular spaces: Wedin’s bound

$A, \tilde{A} \in \mathbb{K}^{I \times J}$ and define $E = \tilde{A} - A$. We fix $1 \leq k \leq r_{\text{max}} = \min\{\#I, \#J\}$ and we consider the SVD

$$A = \begin{pmatrix} U_1 & U_0 \end{pmatrix} \begin{pmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_0 \end{pmatrix} \begin{pmatrix} V_1^H \\ V_0^H \end{pmatrix} = \underbrace{U_1 \Sigma_1 V_1^H}_{:= A_1} + \underbrace{U_0 \Sigma_0 V_0^H}_{:= A_0},$$

where

$$V_1 = [v_1, \ldots, v_k], \quad V_0 = [v_{k+1}, \ldots, v_{r_{\text{max}}}], \quad U_1 = [u_1, \ldots, u_k], \quad U_0 = [u_{k+1}, \ldots, u_{r_{\text{max}}}]$$

and

$$\Sigma_1 = \text{diag}(\sigma_1, \ldots, \sigma_k), \quad \Sigma_0 = \text{diag}(\sigma_{k+1}, \ldots, \sigma_{r_{\text{max}}}).$$
Stability of singular spaces: Wedin’s bound

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$$\Sigma_1 = \text{diag}(\sigma_1, \ldots, \sigma_k), \quad \Sigma_0 = \text{diag}(\sigma_{k+1}, \ldots, \sigma_{r_{\max}}).$$

For the same $k$ we use corresponding notations for an analogous decomposition for $\tilde{A}$ except that all the symbols have $\tilde{\cdot}$ on the top.

It is obvious that

$$A + E = A_1 + A_0 + E = \tilde{A}_1 + \tilde{A}_0 = \tilde{A}. \quad (17)$$
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Let us notice that the range \( A_1 \) is indeed spanned by the orthonormal system \( U_1 \), while range \( A_1^H \) is spanned by \( V_1 \). Similar statements hold for \( \tilde{A}_1 \). We define now two “residual” matrices

\[
R_{11} = A \tilde{V}_1 - \tilde{U}_1 \tilde{\Sigma} \quad \text{and} \quad R_{21} = A^H \tilde{U}_1 - \tilde{V}_1 \tilde{\Sigma}^H.
\]
A connection between $R_{11}$ and $E$ is seen through the rewriting

$$R_{11} = A\tilde{V}_1 - \tilde{U}_1\tilde{\Sigma} = (\tilde{A} - E)\tilde{V}_1 - \tilde{U}_1(\tilde{U}_1^H\tilde{A}\tilde{V}_1) = -E\tilde{V}_1.$$
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We used that $\tilde{\Sigma} = \tilde{U}_1^H\tilde{A}\tilde{V}_1 = \tilde{U}_1^H\tilde{A}_1\tilde{V}_1$ and that $\tilde{U}_1\tilde{U}_1^H$ is the orthogonal projection onto the range of $\text{range} (\tilde{A}_1)$,
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$$R_{21} = -E^H \tilde{U}_1.$$
Theorem (Wedin)

Under the notations so far used in this section, assume that there exist an \( \alpha \geq 0 \) and \( \delta > 0 \) such that

\[
\sigma_k(\tilde{A}) \geq \alpha + \delta \quad \text{and} \quad \sigma_{k+1}(A) \leq \alpha.
\]  

(18)
Theorem (Wedin)

Under the notations so far used in this section, assume that there exist an $\alpha \geq 0$ and $\delta > 0$ such that

$$\sigma_k(\tilde{A}) \geq \alpha + \delta \text{ and } \sigma_{k+1}(A) \leq \alpha.$$  \hfill (18)

Then for every unitarily invariant norm $\| \cdot \|_\star$ (in particular for the spectral and Frobenius norm)

$$\max\{\| \sin \Theta(\tilde{V}_1, V_1)\|_\star, \| \sin \Theta(\tilde{U}_1, U_1)\|_\star\}$$

$$\leq \frac{\max\{\| R_{11} \|_\star, \| R_{21} \|_\star\}}{\delta} \leq \frac{\| E \|_\star}{\delta}.$$
Introduction to probability theory

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In the probabilistic proof of the convergence of the power method for computing singular vectors of a matrix $A$, we used the following random process to define a random point on the sphere: we considered an imaginary player $X$ of darts, with no experience, throwing darts - actually at random - at the target, the two dimensional disk $\Omega = B_r$, the disk of radius $r$ and center $0$. 
Introduction to probability theory

In the probabilistic proof of the convergence of the power method for computing singular vectors of a matrix $A$, we used the following random process to define a random point on the sphere: we considered an imaginary player $X$ of darts, with no experience, throwing darts - actually at random - at the target, the two dimensional disk $\Omega = B_r$, the disk of radius $r$ and center 0. At every point $\omega \in \Omega$ within the target hit by a dart we assign a point on the boundary of the target (the one dimensional sphere $S^1$) $X(\omega) = \frac{\omega}{\|\omega\|_2} \in S^1$. 
Introduction to probability theory

It is also true that the one dimensional sphere $\mathbb{S}^1$ is parametrized by the angle $\theta \in [0, 2\pi)$, hence we can also consider the player $X$ as a map from $\omega \in \Omega$ to $\theta = \arg \frac{\omega}{\|\omega\|_2} \in [0, 2\pi) \subset \mathbb{R}$

$$X : \Omega \to \mathbb{R}.$$
Introduction to probability theory

It is also true that the one dimensional sphere $\mathbb{S}^1$ is parametrized by the angle $\theta \in [0, 2\pi)$, hence we can also consider the player $X$ as a map from $\omega \in \Omega$ to $\theta = \arg \frac{\omega}{\|\omega\|_2} \in [0, 2\pi) \subset \mathbb{R}$

$$X : \Omega \to \mathbb{R}.$$ 

We defined the probability that the very inexperienced player $X$ hits a point $\omega \in B \subset \Omega$ simply as the ratio of the area of $B$ relative to the entire area of the target $\Omega$:

$$\mathbb{P}(\omega \in B) = \frac{\text{Vol}(B)}{\text{Vol}(\Omega)}.$$ 

What is then the probability of the event

$$B = \{\omega \in \Omega : \theta_1 \leq X(\omega) \leq \theta_2\}?$$
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The area of $B$ is the area of the disk sector of vectors with angles between $[\theta_1, \theta_2]$, which is computed by using the polar coordinates:

$$\text{Vol}(B) = \int_{\theta_1}^{\theta_2} \int_0^r sdsd\theta = \frac{r^2(\theta_2 - \theta_1)}{2}. $$
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\]

Hence, recalling that \( \text{Vol}(\Omega) = r^2\pi \), we have

\[
\mathbb{P}(\{ \omega \in \Omega : \theta_1 \leq X(\omega) \leq \theta_2 \}) = \frac{\text{Vol}(B)}{\text{Vol}(\Omega)} = \frac{r^2(\theta_2 - \theta_1)}{2\pi r^2} = \frac{\theta_2 - \theta_1}{2\pi}.
\]
Introduction to probability theory

What is the average outcome of the (random) player?
Introduction to probability theory

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which, for \( N \to \infty \) converges to

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\mathbb{E}X := \int_{\Omega} X(\omega) d\mathbb{P}(\omega) = \frac{1}{2\pi} \int_{0}^{2\pi} \theta d\theta = \frac{(2\pi)^2}{4\pi} = \pi.
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\]

The function \( \phi(\theta) = \frac{1}{2\pi} I_{[0,2\pi]}(\theta) \) is called the probability density of the random player \( X \). In general one computes the quantities

\[
\mathbb{E}g(X) := \int_{\Omega} g(X(\omega)) d\mathbb{P}(\omega) = \int_{\mathbb{R}} g(\theta) \varphi(\theta) d\theta.
\]
If we define \( \Sigma \) the set of all possible events (identified with measurable subsets of \( \Omega \)) the triple \((\Omega, \Sigma, \mathbb{P})\) is called a probability space.
Introduction to probability theory

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As already mentioned above, this particular random variable has a density function $\phi(\theta)$ which allows to compute interesting quantities (such as the probabilities of certain events) by integrals on the real line.
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This relatively intuitive construction is generalized to abstract probability spaces as follows.
Abstract probability theory

Let \((\Omega, \Sigma, \mathbb{P})\) be a probability space, where \(\Sigma\) denotes a \(\sigma\)-algebra (of the events) on the sample space and \(\mathbb{P}\) a probability measure on \((\Omega, \Sigma)\).
Abstract probability theory

Let \((\Omega, \Sigma, \mathbb{P})\) be a probability space, where \(\Sigma\) denotes a \(\sigma\)-algebra (of the events) on the sample space and \(\mathbb{P}\) a probability measure on \((\Omega, \Sigma)\).

The probability of an event \(B \in \Sigma\) is denoted by

\[ \mathbb{P}(B) = \int_B \mathbb{P}(\omega) = \int_\Omega I_B(\omega) \, d\mathbb{P}(\omega) \]

where the characteristic function \(I_B(\omega)\) takes the value 1 if \(\omega \in B\) and 0 otherwise.
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where the characteristic function \(I_B(\omega)\) takes the value 1 if \(\omega \in B\) and 0 otherwise.

The union bound (or Bonferroni’s inequality, or Boole’s inequality) states that for a collection of events \(B_\ell \in \Sigma, \ell = 1, \ldots, n\), we have

\[
\mathbb{P}\left( \bigcup_{\ell=1}^n B_\ell \right) \leq \sum_{\ell=1}^n \mathbb{P}(B_\ell) \quad (19)
\]
A random variable $X$ is a real-valued measurable function on $(\Omega, \Sigma)$.
Random variables

A random variable $X$ is a real-valued measurable function on $(\Omega, \Sigma)$. Recall that $X$ is called measurable if the preimage $X^{-1}(A) = \{\omega \in \Omega : X(\omega) \in A\}$ is contained in $\Sigma$ for all Borel measurable subsets $A \subset \mathbb{R}$. 
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In what follows we will usually not mention the underlying probability space $(\Omega, \Sigma, \mathbb{P})$ when speaking about random variables.
A \textit{random variable} $X$ is a real-valued measurable function on $(\Omega, \Sigma)$. Recall that $X$ is called measurable if the preimage $X^{-1}(A) = \{\omega \in \Omega : X(\omega) \in A\}$ is contained in $\Sigma$ for all Borel measurable subsets $A \subset \mathbb{R}$.

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The \textit{distribution function} $F = F_X$ of $X$ is defined as

$$F(t) = \mathbb{P}(X \leq t), \quad t \in \mathbb{R}. \quad (20)$$
Densities

A random variable $X$ possesses a \textit{probability density function} $\phi: \mathbb{R} \rightarrow \mathbb{R}_+$ if

$$P(a < X \leq b) = \int_a^b \phi(t) \, dt \quad \text{for all } a < b \in \mathbb{R} \quad (21)$$

Then $\phi = \frac{d}{dt} F$. The expectation or mean of a random variable will be denoted by $E_X$. If $X$ has probability density function $\phi$ then for a function $g: \mathbb{R} \rightarrow \mathbb{R}$,

$$E[g(X)] = \int_{\mathbb{R}} g(t) \phi(t) \, dt \quad \text{whenever the integral exists.} \quad (23)$$
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If $X$ has probability density function $\phi$ then for a function $g : \mathbb{R} \to \mathbb{R}$,

$$
\mathbb{E}g(X) = \int_{-\infty}^{+\infty} g(t)\phi(t) \, dt \quad (23)
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Moments

The quantities $\mathbb{E} X^p$, $p > 0$ are called moments of $X$, while $\mathbb{E} |X|^p$ are called absolute moments. (Sometimes we may omit “absolute”.)

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For $1 \leq p < \infty$, $(\mathbb{E} |X|^p)^{\frac{1}{p}}$ defines a norm on the $L^p(\Omega, \mathbb{P})$-space of all $p$-integrable random variables, in particular, the triangle inequality

$$
(\mathbb{E} |X + Y|^p)^{\frac{1}{p}} \leq (\mathbb{E} |X|^p)^{\frac{1}{p}} + (\mathbb{E} |Y|^p)^{\frac{1}{p}}
$$

holds for all $p$-integrable random variables $X, Y$ on $(\Omega, \Sigma, \mathbb{P})$. 

Hoelder’s inequality states that, for random variables $X, Y$ on a common probability space and $p, q \geq 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, we have

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|\mathbb{E} XY| \leq (\mathbb{E} |X|^p)^{\frac{1}{p}} (\mathbb{E} |Y|^q)^{\frac{1}{q}}
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Sequences of random variables and convergence

Let $X_n, n \in \mathbb{N}$, be a sequence of random variables such that $X_n$ converges to $X$ as $n \to \infty$ in the sense that $\lim_{n \to \infty} X_n(\omega) = X(\omega)$ for all $\omega$. 

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*Lebesgue’s dominated convergence theorem* states that if there exists a random variable $Y$ with $\mathbb{E}|Y| < \infty$ such that $|X_n| \leq |Y|$ a.s. then $\lim_{n \to \infty} \mathbb{E}X_n = \mathbb{E}X$. 
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Lebesgue’s dominated convergence theorem has as well an obvious formulation for integrals of sequences of functions.
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Let $f : A \times B \rightarrow C$ be measurable, where $(A, \nu)$ and $(B, \mu)$ are measurable spaces.
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Let $f : A \times B \to C$ be measurable, where $(A, \nu)$ and $(B, \mu)$ are measurable spaces.

If $\int_{A\times B} |f(x, y)| \, d(\nu \otimes \mu)(x, y) < \infty$ then

$$\int_A \left( \int_B f(x, y) \, d\mu(y) \right) \, d\nu(x) = \int_B \left( \int_A f(x, y) \, d\nu(x) \right) \, d\mu(y).$$
Absolute moments can be computed by means of the following formula.

**Proposition**

The absolute moments of a random variable $X$ can be expressed as

$$
\mathbb{E}|X|^p = p \int_0^\infty \mathbb{P}(|X| \geq t) t^{p-1} \, dt, \quad p > 0.
$$
Moment computations and Cavalieri’s formula

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**Corollary**

*For a random variable* $X$ *the expectation satisfies*

$$
\mathbb{E}X = \int_0^\infty \mathbb{P}(X \geq t) \, dt - \int_0^\infty \mathbb{P}(X \leq -t) \, dt.
$$
The function $t \rightarrow \mathbb{P}(|X| \geq t)$ is called the tail of $X$. A simple but often effective tool to estimate the tail by expectations and moments is the Markov inequality.
Tail bounds and Markov inequality

The function $t \rightarrow \mathbb{P}(|X| \geq t)$ is called the

\textit{tail} of $X$. A simple but often effective tool to estimate the tail by expectations and moments is the \textit{Markov inequality}.

\textbf{Theorem}

\textit{Let $X$ be a random variable. Then}

$$
\mathbb{P}(|X| \geq t) \leq \frac{\mathbb{E}|X|}{t} \quad \text{for all } t > 0.
$$
Remark

As an important consequence we note that for $p > 0$

$$
P(|X| \geq t) = P(|X|^p \geq t^p) \leq t^{-p}E|X|^p, \quad \text{for all } t > 0.
$$

The special case $p = 2$ is referred to as the Chebyshev inequality. Similarly, for $\theta > 0$ we obtain

$$
P(e^{\theta X} \geq e^{\theta t}) \leq e^{-\theta t}Ee^{\theta X} \quad \text{for all } t \in \mathbb{R}.
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The function $\theta \mapsto Ee^{\theta X}$ is usually called the Laplace transform or the moment generating function of $X$. 

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Normal distributions

A *normal distributed* random variable or *Gaussian random variable* $X$ has probability density function

$$
\psi(t) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(t - \mu)^2}{2\sigma^2}\right).
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Its distribution is often denoted by $N(\mu, \sigma)$. 
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It has mean $\mathbb{E}X = \mu$ and variance $\mathbb{E}(X - \mu)^2 = \sigma^2$.

A standard Gaussian random variable (or standard normal or simply standard Gaussian), usually denoted $g$, is a Gaussian random variable with $\mathbb{E}g = 0$ and $\mathbb{E}g^2 = 1$. 
Random vectors

A *random vector* $\mathbf{X} = [X_1, \ldots, X_n]^\top \in \mathbb{R}^n$ is a collection of $n$ random variables on a common probability space $(\Omega, \Sigma, \mathbb{P})$. 
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Its expectation is the vector $\mathbb{E}\mathbf{X} = [\mathbb{E}X_1, \ldots, \mathbb{E}X_n]^\top \in \mathbb{R}^n$, while its joint distribution function is defined as

$$F(t_1, \ldots, t_n) = \mathbb{P}(X_1 \leq t_1, \ldots, X_n \leq t_n), \quad t_1, \ldots, t_n \in \mathbb{R}.$$
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Similarly to the univariate case, the random vector \( \mathbf{X} \) has a joint probability density if there exists a function \( \phi: \mathbb{R}^n \to [0,1] \) such that for any measurable domain \( D \subset \mathbb{R}^n \)

\[
\mathbb{P}(\mathbf{X} \in D) = \int_D \phi(t_1, \ldots, t_n) \, dt_1 \cdots dt_n.
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$$\mathbb{P}(\mathbf{X} \in D) = \int_D \phi(t_1, \ldots, t_n) \, dt_1 \cdots dt_n.$$ 

A complex random vector $\mathbf{Z} = \mathbf{X} + i\mathbf{Y} \in \mathbb{C}^n$ is a special case of a $2n$-dimensional real random vector $(\mathbf{X}, \mathbf{Y}) \in \mathbb{R}^{2n}$. 
Independence

A collection of random variables $X_1, \ldots, X_n$ is (stochastically) independent if, for all $t_1, \ldots, t_n \in \mathbb{R}$,

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\mathbb{P}(X_1 \leq t_1, \ldots, X_n \leq t_n) = \prod_{\ell=1}^{n} \mathbb{P}(X_\ell \leq t_\ell).
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For independent random variables, we have

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\mathbb{E} \left[ \prod_{\ell=1}^{n} X\ell \right] = \prod_{\ell=1}^{n} \mathbb{E}[X\ell].
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$$

If they have a joint probability density function $\phi$ then the latter factorizes as

$$
\phi(t_1, \ldots, t_n) = \phi_1(t_1) \times \cdots \times \phi_n(t_n)
$$

where the $\phi_1, \ldots, \phi_n$ are the probability density functions of $X_1, \ldots, X_n$. 
Independence

In general, a collection $\mathbf{X}_1 \in \mathbb{R}^{m_1}, \ldots, \mathbf{X}_m \in \mathbb{R}^{m_m}$ of random vectors are independent if for any collection of measurable sets $A_\ell \subset \mathbb{R}^{n_\ell}$, $\ell \in [m]$,

$$
P(\mathbf{X}_1 \in A_1, \ldots, \mathbf{X}_m \in A_m) = \prod_{\ell=1}^m P(\mathbf{X}_\ell \in A_\ell).
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In general, a collection $\mathbf{X}_1 \in \mathbb{R}^{n_1}, \ldots, \mathbf{X}_m \in \mathbb{R}^{n_m}$ of random vectors are independent if for any collection of measurable sets $A_\ell \subset \mathbb{R}^{n_\ell}, \ell \in [m],$

$$\mathbb{P}(\mathbf{X}_1 \in A_1, \ldots, \mathbf{X}_m \in A_m) = \prod_{\ell=1}^{m} \mathbb{P}(\mathbf{X}_\ell \in A_\ell).$$

If furthermore $f_\ell : \mathbb{R}^{n_\ell} \rightarrow \mathbb{R}^{N_\ell}, \ell = 1, \ldots, m,$ are measurable functions then also the random vectors $f_1(\mathbf{X}_1), \ldots, f_m(\mathbf{X}_m)$ are independent.
Independence

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If furthermore $f_\ell: \mathbb{R}^{n_\ell} \to \mathbb{R}^{N_\ell}$, $\ell = 1, \ldots, m$, are measurable functions then also the random vectors $f_1(X_1), \ldots, f_m(X_m)$ are independent.

A collection $X_1, \ldots, X_m \in \mathbb{R}^n$ of independent random vectors that all have the same distribution is called \textit{independent identically distributed} (i.i.d.).

A random vector $X'$ will be called an independent copy of $X$ if $X$ and $X'$ are independent and have the same distribution.
The sum $X + Y$ of two independent random variables $X$, $Y$ having probability density functions $\phi_X$, $\phi_Y$, has probability density function $\phi_{X+Y}$ given by the convolution

$$\phi_{X+Y}(t) = (\phi_X * \phi_Y)(t) = \int_{-\infty}^{+\infty} \phi_X(u) \phi_Y(t-u) \, du \quad (27)$$
Fubini’s theorem for expectations

The theorem takes the following form. Let \( X, Y \in \mathbb{R}^n \) be two independent random vectors (or simply random variables) and \( f : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R} \) be a measurable function such that 
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E|f(X, Y)| < \infty.
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$$f_1 : \mathbb{R}^n \to \mathbb{R}, \quad f_1(x) = \mathbb{E}f(x, \mathbf{Y}), \quad f_2 : \mathbb{R}^n \to \mathbb{R}, \quad f_2(y) = \mathbb{E}f(\mathbf{X}, y)$$

are measurable, $\mathbb{E}|f_1(\mathbf{X})| < \infty$ and $\mathbb{E}|f_2(\mathbf{Y})| < \infty$ and

$$\mathbb{E}f_1(\mathbf{X}) = \mathbb{E}f_2(\mathbf{Y}) = \mathbb{E}f(\mathbf{X}, \mathbf{Y}).$$

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\]

The random variable \( f_1(\mathbf{X}) \) is also called conditional expectation or expectation conditional on \( \mathbf{X} \) and will sometimes be denoted by \( \mathbb{E}_Y f(\mathbf{X}, \mathbf{Y}) \).
Gaussian vectors

A random vector \( \mathbf{g} \in \mathbb{R}^n \) is called a standard Gaussian vector if its components are independent standard normal distributed random variables.
Gaussian vectors

A random vector $g \in \mathbb{R}^n$ is called a standard Gaussian vector if its components are independent standard normal distributed random variables. More generally, a random vector $X \in \mathbb{R}^n$ is said to be a Gaussian vector or multivariate normal distributed if there exists a matrix $A \in \mathbb{R}^{n \times k}$ such that $X = Ag + \mu$, where $g \in \mathbb{R}^k$ is a standard Gaussian vector and $\mu \in \mathbb{R}^n$ is the mean of $X$. 

The matrix $\Sigma = AA^T$ is then the covariance matrix of $X$, i.e.,

$$
\Sigma = \mathbb{E}[(X - \mu)(X - \mu)^T].
$$

If $\Sigma$ is non-degenerate, i.e., $\Sigma$ is positive definite, then $X$ has a joint probability density function of the form

$$
\psi_{\mathbb{R}^n}(x) = \frac{1}{\sqrt{2\pi n \det(\Sigma)}} \exp\left(-\frac{1}{2} x^T \Sigma^{-1} x\right).
$$

In the degenerate case when $\Sigma$ is not invertible $X$ does not have a density. It is easily deduced from the density that a rotated standard Gaussian $Ug$, where $U$ is an orthogonal matrix, has the same distribution as $g$ itself.
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The matrix \( \Sigma = \mathbf{A} \mathbf{A}^T \) is then the covariance matrix of \( \mathbf{X} \), i.e., \( \Sigma = \mathbb{E}(\mathbf{X} - \mu)(\mathbf{X} - \mu)^\top \). If \( \Sigma \) is non-degenerate, i.e., \( \Sigma \) is positive definite, then \( \mathbf{X} \) has a joint probability density function of the form

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\psi(\mathbf{x}) = \frac{1}{(2\pi)^{n/2} \sqrt{\det(\Sigma)}} \exp\left(-\frac{1}{2} \langle \mathbf{x} - \mu, \Sigma^{-1}(\mathbf{x} - \mu) \rangle \right)
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Gaussian vectors

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It is easily deduced from the density that a rotated standard Gaussian $Ug$, where $U$ is an orthogonal matrix, has the same distribution as $g$ itself.
If $X_1, \ldots, X_n$ are independent and normal distributed random variables with means $\mu_1, \ldots, \mu_n$ and variances $\sigma_1^2, \ldots, \sigma_n^2$ then $X = [X_1, \ldots, X_n]^\top$ has a multivariate normal distribution and its sum $Z = \sum_{\ell=1}^n X_\ell$ has the univariate normal distribution with mean $\mu = \sum_{\ell=1}^n \mu_\ell$ and variance $\sigma^2 = \sum_{\ell=1}^n \sigma_\ell^2$, as can be calculated from (27).
Jensen’s inequality reads as follows.

**Theorem**

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function, and let $X \in \mathbb{R}^n$ be a random vector. Then

$$f(\mathbb{E}X) \leq \mathbb{E}f(X).$$

Note that $-f$ is convex if $f$ is concave, so that for concave functions $f$, Jensen’s inequality reads

$$\mathbb{E}f(X) \leq f(\mathbb{E}X).$$  \hspace{1cm} (29)
Cramér’s Theorem

We often encounter sums of independent mean zero random variables. Deviation inequalities bound the tail of such sums.
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\theta \mapsto \mathbb{E} \exp(\theta X),
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for all $\theta \in \mathbb{R}$ whenever the expectation on the right hand side is well-defined.
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Its logarithm is the *cumulant generating function*

$$C_X(\theta) = \ln \mathbb{E} \exp(\theta X).$$

With the help of these definitions we can formulate Cramér’s theorem.
Cramér’s Theorem

Theorem
Let $X_1, \ldots, X_M$ be a sequence of independent (real-valued) random variables, with cumulant generating functions $C_{X_\ell}$, $\ell \in [M] := \{1, \ldots, M\}$. 
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\[
\mathbb{P}\left( \sum_{\ell=1}^{M} X_\ell \geq t \right) \leq \exp \left( \inf_{\theta > 0} \left\{ -\theta t + \sum_{\ell=1}^{M} C_{X_\ell}(\theta) \right\} \right).
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$$

**Remark**

The function

$$
t \mapsto \inf_{\theta > 0} \left\{ -\theta t + \sum_{\ell=1}^{M} X_{X_\ell}(\theta) \right\}
$$

appearing in the exponential is closely connected to a convex conjugate function appearing in convex analysis.
Hoeffdings’ inequality

Theorem

Let \( X_1, \ldots, X_M \) be a sequence of independent random variables such that \( \mathbb{E}X_\ell = 0 \) and \( |X_\ell| \leq B_\ell \) almost surely, \( \ell \in [M] \).
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$$

and consequently,

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\mathbb{P} \left( \left| \sum_{\ell=1}^{M} X_\ell \right| \geq t \right) \leq 2 \exp \left( - \frac{t^2}{2 \sum_{\ell=1}^{M} B_\ell^2} \right). \quad (30)
$$
Bernstein inequalities

Theorem
Let $X_1, \ldots, X_M$ be independent mean zero random variables such that, for all integers $n \geq 2$,

\[ \mathbb{E}|X_\ell|^n \leq \frac{n!}{2} R^{n-2} \sigma_\ell^2 \quad \text{for all } \ell \in [M] \tag{31} \]

for some constants $R > 0$ and $\sigma_\ell > 0$, $\ell \in [M]$. 
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$$\mathbb{E}|X_\ell|^n \leq \frac{n!}{2} R^{n-2} \sigma_\ell^2 \quad \text{for all } \ell \in [M]$$  \hspace{1cm} (31)

for some constants $R > 0$ and $\sigma_\ell > 0$, $\ell \in [M]$. Then, for all $t > 0$,

$$\mathbb{P}\left(\left| \sum_{\ell=1}^M X_\ell \right| \geq t \right) \leq 2 \exp\left(-\frac{t^2}{2(\sigma^2 + R t)}\right),$$  \hspace{1cm} (32)

where $\sigma^2 := \sum_{\ell=1}^M \sigma_\ell^2$. 

Before providing the proof we give two consequences. The first is the Bernstein inequality for bounded random variables.

**Corollary**

Let $X_1, \ldots, X_M$ be independent random variables with zero mean such that $|X_\ell| < K$ almost surely, for $\ell \in [M]$ and some constant $K > 0$. Further assume $\mathbb{E}|X_\ell|^2 \leq \sigma_\ell^2$ for constants $\sigma_\ell > 0$, $\ell \in [M]$. Then, for all $t > 0$,

$$
\mathbb{P} \left( \left| \sum_{\ell=1}^{M} X_\ell \right| \geq t \right) \leq 2 \exp \left( -\frac{t^2}{2(\sigma^2 + \frac{1}{3}Kt)} \right), \quad (33)
$$

where $\sigma^2 := \sum_{\ell=1}^{M} \sigma_\ell^2$. 


Bernstein inequalities

As a second consequence, we present the Bernstein inequality for subexponential random variables.

**Corollary**

Let $X_1, \ldots, X_M$ be independent mean zero subexponential random variables, that is, $\mathbb{P}(\mid X_\ell \mid \geq t) \leq \beta e^{-\kappa t}$ for some constants $\beta, \kappa > 0$ for all $t > 0$, $\ell \in [M]$. Then

$$\mathbb{P}\left( \left\mid \sum_{\ell=1}^{M} X_\ell \right\mid \geq t \right) \leq 2 \exp\left( - \frac{(\kappa t)^2}{2(2\beta M + \kappa t)} \right), \tag{34}$$
The Johnson-Lindenstrauss Lemma

Theorem (Johnson-Lindenstrauss Lemma)

For any $0 < \varepsilon < 1$ and any integer $n$, let $k$ be a positive integer and such that

$$k \geq 2\beta (\varepsilon^2/2 - \varepsilon^3/3)^{-1} \ln n,$$

for some $\beta \geq 2$. 
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for some $\beta \geq 2$. Then for any set $\mathcal{P}$ of $n$ points in $\mathbb{R}^d$, there is a map $f : \mathbb{R}^d \rightarrow \mathbb{R}^k$ such that for all $v, w \in \mathcal{P}$

$$(1 - \varepsilon)\|v - w\|_2^2 \leq \|f(v) - f(w)\|_2^2 \leq (1 + \varepsilon)\|v - w\|_2^2 \quad (35)$$

Furthermore, this map can be generated at random with probability $1 - n^{2\beta} n^2 \beta$. Let us comment on the probability: choosing $\beta \approx 2$ leads to a very favorable probability $1 - n^{2\beta} n^2 \beta$ as soon as $n$ is large enough.
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Furthermore, this map can be generated at random with probability $1 - (n^{2-\beta} - n^{1-\beta})$.

Let us comment on the probability: choosing $\beta \gg 2$ leads to a very favorable probability $1 - (n^{2-\beta} - n^{1-\beta}) \approx 1$ as soon as $n$ is large enough.
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To clarify the impact of this result, consider a cloud of $n = 10^7$ points (ten million points) in dimension $d = 10^6$ (one million).
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Then there exists a map $f$ generated at random with probability close to 1, which allows us to project the 10 million points in dimension $k = \mathcal{O}(\varepsilon^{-2} \log 10^6) = \mathcal{O}(6\varepsilon^{-2})$, preserving the distances within the point cloud with a maximal distortion of $\mathcal{O}(\varepsilon)$. 
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Then there exists a map $f$ generated at random with probability close to 1, which allows us to project the 10 million points in dimension $k = O(\varepsilon^{-2} \log 10^6) = O(6\varepsilon^{-2})$, preserving the distances within the point cloud with a maximal distortion of $O(\varepsilon)$.

For example, for $\varepsilon = 0.1$ we obtain $k$ of the order of a thousand, while the original dimension $d$ was one million. Hence, we obtain a quasi-isometrical dimensionality reduction of 3-4 orders of magnitude.
One proof of the lemma takes $f$ to be a suitable multiple of the orthogonal projection onto a random subspace of dimension $k$ in $\mathbb{R}^d$ (Gupta, Dasgupta).
Projection onto random subspaces

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In this case $f$ would be linear and representable by a matrix $A$. 

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Projection onto random subspaces

One proof of the lemma takes $f$ to be a suitable multiple of the orthogonal projection onto a random subspace of dimension $k$ in $\mathbb{R}^d$ (Gupta, Dasgupta).

In this case $f$ would be linear and representable by a matrix $A$.

Then, by considering unit vectors $z = v - w/\|v - w\|_2$, we obtain that (35) is equivalent to

$$(1 - \varepsilon) \leq \|Az\|^2 \leq (1 + \varepsilon).$$

or

$$\|\|Az\|^2 - 1\| \leq \varepsilon.$$
... or a random point onto a fixed subspace

Hence the aim is essentially to estimate the length of a unit vector in $\mathbb{R}^d$ when it is projected onto a random $k$-dimensional subspace.
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However, this length has the same distribution as the length of a random unit vector projected down onto a fixed $k$-dimensional subspace.
... or a random point onto a fixed subspace

Hence the aim is essentially to estimate the length of a unit vector in $\mathbb{R}^d$ when it is projected onto a random $k$-dimensional subspace.

However, this length has the same distribution as the length of a random unit vector projected down onto a fixed $k$-dimensional subspace.

Here we take this subspace to be the space spanned by the first $k$ coordinate vectors, for simplicity.
Gaussianly distributed vectors

We recall that $X \sim N(0, 1)$ if its density is

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\psi(x) = \frac{1}{\sqrt{2\pi}} \exp(-x^2/2).
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Let $X_1, \ldots, X_d$ be $d$-independent Gaussian $N(0, 1)$ random variables, and let $Y = \|X\|_2^{-1}(X_1, \ldots, X_d)$. 
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It is easy to see that $Y$ is a point chosen uniformly at random from the surface of the $d$-dimensional sphere $\mathbb{S}^{d-1}$ (exercise!).

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It is easy to see that $Y$ is a point chosen uniformly at random from the surface of the $d$-dimensional sphere $\mathbb{S}^{d-1}$ (exercise!).

Let the vector $Z \in \mathbb{R}^k$ be the projection of $Y$ onto its first $k$ coordinates, and let $L = \|Z\|_2^2$. Clearly the expected squared length of $Z$ is $\mu = \mathbb{E}[\|Z\|_2^2] = k/d$ (exercise!).
Concentration inequalities

Lemma
Let $k < d$. 
Concentration inequalities

Lemma

Let $k < d$. Then,

(a) If $\alpha < 1$, then

$$
\mathbb{P}
\left( L \leq \alpha \frac{k}{d} \right)
\leq 
\alpha^{k/2} \left( 1 + \frac{(1 - \alpha)k}{d - k} \right)^{(d-k)/2}
\leq 
\exp \left( \frac{k}{2} (1 - \alpha + \ln \alpha) \right).
$$
Concentration inequalities

Lemma

Let \( k < d \). Then,

(a) If \( \alpha < 1 \), then

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\mathbb{P} \left( L \leq \alpha \frac{k}{d} \right) \leq \alpha^{k/2} \left( 1 + \frac{(1 - \alpha)k}{d - k} \right)^{(d-k)/2} \\
\leq \exp \left( \frac{k}{2} (1 - \alpha + \ln \alpha) \right).
\]

(b) If \( \alpha > 1 \), then

\[
\mathbb{P} \left( L \geq \alpha \frac{k}{d} \right) \leq \alpha^{k/2} \left( 1 + \frac{(1 - \alpha)k}{d - k} \right)^{(d-k)/2} \\
\leq \exp \left( \frac{k}{2} (1 - \alpha + \ln \alpha) \right).
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Proof techniques

The proof of this lemma uses by now standard techniques applied for proving large deviation bounds on sums of random variables such as the ones we used for proving the Hoeffding and Bernstein inequalities.
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The proof of the Johnson-Lindenstrauss Lemma - as developed at the blackboard - follows by union bound over the application of the concentration inequalities of the Lemma.
Theorem (Johnson-Lindenstrauss Lemma)

For any $0 < \varepsilon < 1/2$ and any integer $n$, let $k$ be a positive integer and such that

$$k \geq \beta \varepsilon^{-2} \ln n.$$ 

Then for any set $P$ of $n$ points in $\mathbb{R}^d$, there is a map $f : \mathbb{R}^d \to \mathbb{R}^k$ such that for all $v, w \in P$

$$(1 - \varepsilon)\|v - w\|_2^2 \leq \|f(v) - f(w)\|_2^2 \leq (1 + \varepsilon)\|v - w\|_2^2 \quad (36)$$

Furthermore, this map can be generated at random with probability $1 - (n^{2-\beta(1-\varepsilon)} - n^{1-\beta(1-\varepsilon)})$. 
Concentration inequalities for Gaussian matrices

**Lemma**

Let \( x \in \mathbb{R}^d \). Assume that the entries of \( A \in \mathbb{R}^{k \times d} \) are sampled independently and identically distributed according to \( N(0, 1) \). Then

\[
P \left( \left\| \frac{1}{\sqrt{k}} Ax \right\|_2^2 - \|x\|_2^2 > \varepsilon \|x\|_2^2 \right) \leq 2e^{-(\varepsilon^2 - \varepsilon^3)k/4},
\]

or, equivalently,

\[
P \left( (1 - \varepsilon)\|x\|_2^2 \leq \left\| \frac{1}{\sqrt{k}} Ax \right\|_2^2 \leq (1 + \varepsilon)\|x\|_2^2 \right) \geq 1 - 2e^{-(\varepsilon^2 - \varepsilon^3)k/4}.
\]
Nothing special in Gaussian matrices

**Lemma**

Let $x \in \mathbb{R}^d$. Assume that the entries of $A \in \mathbb{R}^{k \times d}$ are either $+1$ or $-1$ with probability $1/2$. Then

$$
P \left( \left\| \frac{1}{\sqrt{k}} Ax \right\|_2^2 - \|x\|_2^2 > \varepsilon \|x\|_2^2 \right) \leq 2e^{-\left(\varepsilon^2 - \varepsilon^3\right)k/4},$$

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Scalar product preservation

Corollary

Let $\mathcal{P}$ be a set points in the unit ball $B(1) \subset \mathbb{R}^d$ and that $f : \mathbb{R}^d \to \mathbb{R}^k$ is a linear function such that

\[(1 - \varepsilon)\|u \pm v\|_2^2 \leq \|f(u \pm v)\|_2^2 \leq (1 + \varepsilon)^2\|u \pm v\|_2^2, \quad (37)\]

for all $u, v \in \mathcal{P}$. Then

$$|\langle u, v \rangle - \langle f(u), f(v) \rangle| \leq \varepsilon,$$

for $u, v \in \mathcal{P}$. 
Scalar product preservation

Corollary

Let $u, v$ be two points in the unit ball $B(1) \subset \mathbb{R}^d$, $A \in \mathbb{R}^{k \times d}$ is a matrix with random entries such that, for a given $x \in \mathbb{R}^d$

$$\mathbb{P} \left( (1 - \varepsilon) \|x\|_2^2 \leq \left\| \frac{1}{\sqrt{k}} Ax \right\|_2^2 \leq (1 + \varepsilon) \|x\|_2^2 \right) \geq 1 - 2e^{-(\varepsilon^2 - \varepsilon^3)k/4}.$$ 

(For instance $A$ is a matrix with Gaussian random entries or with entries $\pm 1$ with probability $1/2$.) Then, with probability at least $1 - 4e^{-(\varepsilon^2 - \varepsilon^3)k/4}$

$$|\langle u, v \rangle - \langle f(u), f(v) \rangle| \leq \varepsilon,$$

for $u, v \in \mathcal{P}$. 
Convex sets

Definition
A subset $K \subset \mathbb{R}^N$ is called convex if for all $x, z \in K$ the line segment connecting $x, z$ is also completely contained in $K$, that is,

$$tx + (1 - t)z \in K \quad \text{for all } t \in [0, 1].$$
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\]

It is straightforward to check that a set \( K \subset \mathbb{R}^N \) is convex if and only if for all \( x_1, \ldots, x_n \in K \) and \( t_1, \ldots, t_n \geq 0 \) such that \( \sum_{j=1}^{n} t_j = 1 \) the convex combination \( \sum_{j=1}^{n} t_j x_j \) is also contained in \( K \).
Convex sets

Definition
Let $T \in \mathbb{R}^N$ be a set. Its convex hull $\text{conv}(T)$ is the smallest convex set containing $T$. 
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It is well-known that the convex hull of \( T \) consists of the convex combinations of \( T \),

\[
\text{conv}(T) = \left\{ \sum_{j} t_j x_j : t_j \geq 0, \sum_{j} t_j = 1, x_j \in T \right\}.
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$$\text{conv}(T) = \left\{ \sum_j t_j x_j : t_j \geq 0, \sum_j t_j = 1, x_j \in T \right\}.$$

Simple examples of convex sets include subspaces, affine spaces, half spaces, polygons or norm balls $B(x, R)$. The intersection of convex sets is again convex.
Cones

Definition
A $K \subset \mathbb{R}^N$ is called a cone if for all $x \in K$ and all $t \geq 0$ also $tx$ is contained in $K$. If, in addition, $K$ is convex, then $K$ is called a convex cone.
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Obviously, the zero vector is contained in every cone. A set $K$ is a convex cone if for all $x, z \in K$ and $t, s \geq 0$ also $sx + tz$ is contained in $K$. 

Simple examples of convex cones include the positive orthant $\mathbb{R}^N_+$: $x \geq 0$ for all $i \in \mathbb{R}$, or the set of positive semidefinite matrices in $\mathbb{R}^{N \times N}$. Another important example of a convex cone is the second order cone $\mathbb{S}_+$: $\sum_{i=1}^{N-1} x^2_i \leq x_N^2$. (38)
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Simple examples of convex cones include the positive orthant $\mathbb{R}^N_+ = \{x \in \mathbb{R}^N : x_i \geq 0 \text{ for all } i \in [N]\}$ or the set of positive semidefinite matrices in $\mathbb{R}^N \times \mathbb{R}^N$. Another important example of a convex cone is the second order cone

$$\left\{ x \in \mathbb{R}^{N+1} : \sqrt{\sum_{j=1}^{N} x_j^2} \leq x_{N+1} \right\}. \quad (38)$$
Dual cones

For a cone $K \subset \mathbb{R}^N$, its dual cone $K^*$ is defined via

$$K^* = \{ z \in \mathbb{R}^N : \langle x, z \rangle \geq 0 \text{ for all } x \in K \}. \quad (39)$$
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As the intersection of half-spaces, $K^*$ is closed and convex, and it is straight-forward to check that $K^*$ is again a cone.
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If $K$ is a closed cone then $K^{**} = K$. 
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If $K$ is a closed cone then $K^{**} = K$.

Moreover, if $H, K \subset \mathbb{R}^N$ are cones such that $H \subset K$ then $K^* \subset H^*$. 

As an example, the dual cone of the positive orthant $\mathbb{R}^N_+$ is $\mathbb{R}^N_+$ itself, i.e. it is self-dual.
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As an example, the dual cone of the positive orthant $\mathbb{R}_+^N$ is $\mathbb{R}_+^N$ itself, i.e. it is self-dual.
Polar cones

Note that the dual cone is closely related to the polar cone, which is defined by

$$K^\circ = \{ z \in \mathbb{R}^N : \langle x, z \rangle \leq 0 \text{ for all } x \in K \} = -K^\star. \quad (40)$$
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The conic hull \( \text{cone}(T) \) of a set \( T \subset \mathbb{R}^N \) is the smallest convex cone containing \( T \).
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The conic hull \( \text{cone}(T) \) of a set \( T \subset \mathbb{R}^N \) is the smallest convex cone containing \( T \).

It can be described as

\[ \text{cone}(T) = \left\{ \sum_j t_j x_j : t_j \geq 0, \ x_j \in T \right\}. \]  (41)
Convex sets can be separated by hyper-planes as stated next.

**Theorem**

Let $K_1, K_2 \subset \mathbb{R}^N$ be convex sets whose interiors have empty intersection. Then there exists a vector $w \in \mathbb{R}^N$ and a scalar $\lambda$ such that

$$K_1 \subset \{ x \in \mathbb{R}^N : \langle x, w \rangle \leq \lambda \},$$

$$K_2 \subset \{ x \in \mathbb{R}^N : \langle x, w \rangle \geq \lambda \}.$$
Definition

Let $K \subset \mathbb{R}^N$ be a convex set. A point $x \in K$ is called extreme point of $K$ if $x = tw + (1 - t)z$ for $w, z \in K$ and $t \in (0, 1)$ implies $x = w = z$. 

Theorem

A compact convex set is the convex hull of its extreme points. If $K$ is a polygon then its extreme points are the zero-dimensional faces of $K$, and the above statement is clearly intuitive.
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Extremal points

Definition
Let \( K \subseteq \mathbb{R}^N \) be a convex set. A point \( x \in K \) is called extreme point of \( K \) if
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x = t w + (1 - t) z \quad \text{for} \quad w, z \in K \quad \text{and} \quad t \in (0, 1)
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Proper functions

We work with extended valued functions $F: \mathbb{R}^N \to (-\infty, \infty]$.
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We work with extended valued functions $F : \mathbb{R}^N \to (-\infty, \infty]$. Sometimes we also consider an additional extension of the values to $-\infty$. Addition, multiplication and inequalities in $(-\infty, \infty]$ are understood in the “natural” sense; for instance, $x + \infty = \infty$ for all $x \in \mathbb{R}$. 
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The domain of an extended-valued function $F$ is defined as

$$\text{dom}(F) = \{ x \in \mathbb{R}^N : F(x) \neq \infty \}. $$
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A function with $\text{dom}(F) \neq \emptyset$ is called proper.

A function $F : K \rightarrow \mathbb{R}$ on a subset $K \subset \mathbb{R}^N$ can be extended to an extended valued function by setting $F(x) = \infty$ for $x \notin K$. Then clearly $\text{dom}(F) = K$, and we call this extension the canonical one.
Convex functions

Definition
An extended valued function $F : \mathbb{R}^N \to (-\infty, \infty]$ is called convex if for all $x, z \in \mathbb{R}^N$ and $t \in [0, 1]$,

$$F(tx + (1 - t)z) \leq tF(x) + (1 - t)F(z).$$  \hspace{1cm} (42)
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$F$ is called strictly convex if above inequality is strict for $x \neq z$ and $t \in (0, 1).$
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$F$ is called strongly convex with parameter $\gamma > 0$ if for all $x, z \in \mathbb{R}^N$ and $t \in [0, 1]$

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A function $F : \mathbb{R}^N \to [-\infty, \infty)$ is called (strictly, strongly) concave if $-F$ is (strictly, strongly) convex.
Convex functions

Obviously, a strongly convex function is strictly convex.
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The domain of a convex function is convex, and a function $F : K \rightarrow \mathbb{R}^N$ on a convex subset $K \subset \mathbb{R}^N$ is called convex if its canonical extension to $\mathbb{R}^N$ is convex.
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A function $F$ is convex if and only if its epigraph

$$\text{epi}(F) = \{(x, r) : r \geq F(x)\} \subset \mathbb{R}^N \times \mathbb{R}$$

is a convex set.
Smooth convex functions

**Proposition**

Let $F : \mathbb{R}^N \rightarrow \mathbb{R}$ be differentiable.

1. $F$ is convex if and only if for all $x, y \in \mathbb{R}^N$, $F_{\lambda x + (1-\lambda)y} \leq \lambda F(x) + (1-\lambda)F(y)$, where the gradient is defined as $\nabla F_{\lambda x + (1-\lambda)y} = \lambda \nabla F(x) + (1-\lambda) \nabla F(y)$.

2. $F$ is strongly convex with parameter $\gamma > 0$ if and only if for all $x, y \in \mathbb{R}^N$, $F_{\lambda x + (1-\lambda)y} \leq \lambda F(x) + (1-\lambda)F(y) - \frac{\gamma}{2} \lambda (1-\lambda) \| x - y \|^2$.

3. Assume that $F$ is twice differentiable. Then $F$ is convex if and only if $\nabla^2 F_{\mathbb{R}^N} \succeq 0$ for all $x \in \mathbb{R}^N$, where $\nabla^2 F$ is the Hessian of $F$. 
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where the gradient is defined as $\nabla F(y) = (\partial_{y_1} F(y), \ldots, \partial_{y_n} F(y))^T$. 

2. $F$ is strongly convex with parameter $\gamma > 0$ if and only if for all $x, y \in \mathbb{R}^N$

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Composition of convex functions

Proposition

1. Let $F, G$ be convex functions on $\mathbb{R}^N$. Then, for $\alpha, \beta \geq 0$ the function $\alpha F + \beta G$ is convex.
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1. Let $F, G$ be convex functions on $\mathbb{R}^N$. Then, for $\alpha, \beta \geq 0$ the function $\alpha F + \beta G$ is convex.

2. Let $F : \mathbb{R} \rightarrow \mathbb{R}$ be convex and nondecreasing, and $G : \mathbb{R}^N \rightarrow \mathbb{R}$ be convex. Then $H(x) = F(G(x))$ is convex.
Examples of convex functions

- Every norm $\| \cdot \|$ on $\mathbb{R}^N$ is convex by triangle inequality and homogeneity.
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- The \( \ell_p \)-norms are strictly convex for \( 1 < p < \infty \) and not strictly convex for \( p = 1, \infty \).
- For \( A \in \mathbb{R}^{N \times N} \) positive semidefinite, the function \( F(x) = x^* A x \) is convex. It is strictly convex if \( A \) is positive definite.
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- The \( \ell_p \)-norms are strictly convex for \( 1 < p < \infty \) and not strictly convex for \( p = 1, \infty \).
- For \( A \in \mathbb{R}^{N \times N} \) positive semidefinite, the function \( F(x) = x^* A x \) is convex. It is strictly convex if \( A \) is positive definite.
- For a convex set \( K \) the characteristic function

\[
\chi_K(x) = \begin{cases} 
0 & x \in K, \\
\infty & \text{else}
\end{cases}
\]  

is convex.
Convex functions, continuity, and lower-semicontinuity

Proposition

Let $F : \mathbb{R}^N \to \mathbb{R}$ be a convex function. Then $F$ is continuous on $\mathbb{R}^N$. 

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Treatment of extended valued functions requires the concept of lower semicontinuity.

Definition
A function $F : \mathbb{R}^N \to (-\infty, \infty]$ is called lower semicontinuous if for all $x \in \mathbb{R}^N$ and every sequence $(x_j)_{j \in \mathbb{N}} \subset \mathbb{R}^N$ converging to $x$ it holds

$$\liminf_{j \to \infty} F(x_j) \geq F(x).$$
Clearly, a continuous function $F : \mathbb{R}^N \rightarrow \mathbb{R}$ is lower semicontinuous.
Clearly, a continuous function $F: \mathbb{R}^N \to \mathbb{R}$ is lower semicontinuous.

A nontrivial example is the characteristic function $\chi_K$ of a proper subset $K \subset \mathbb{R}^N$. Clearly, $\chi_K$ is not continuous, but it is lower semicontinuous if and only if $K$ is closed.
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A function is lower semicontinuous if and only if its epigraph is closed.
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A function is lower semicontinuous if and only if its epigraph is closed.

We remark that the notion of lower semicontinuity is particularly useful in infinite-dimensional Hilbert spaces, where for instance the norm $\| \cdot \|$ is not continuous with respect to the weak topology but is still lower semicontinuous with respect to the weak topology.
Convex functions and minimization

Convex functions have nice properties related to minimization.
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A (global) minimum (or minimizer) of a function \(F: \mathbb{R}^N \rightarrow (-\infty, \infty]\) is a point \(x \in \mathbb{R}^N\) satisfying \(F(x) \leq F(y)\) for all \(y \in \mathbb{R}^N\).

A local minimum of \(F\) is a point \(x \in \mathbb{R}^N\) such that there exists \(\epsilon > 0\) and \(F(x) \leq F(y)\) for all \(y\) satisfying \(\|x - y\|_2 \leq \epsilon\).

Proposition

Let \(F: \mathbb{R}^N \rightarrow (-\infty, \infty]\) be convex.

1. A local minimum of \(F\) is a global minimum.
2. The set of minima of \(F\) is convex.
3. If \(F\) is strictly convex then the minimum is unique.

The fact that local minima of convex functions are automatically global minima is the essential reason why efficient optimization methods are available for convex optimization problems.
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Proposition

Let $F : \mathbb{R}^N \to (-\infty, \infty]$ be convex.

1. A local minimum of $F$ is a global minimum.
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The fact that local minima of convex functions are automatically global minima is the essential reason why efficient optimization methods are available for convex optimization problems.
Convex functions and minimization

Convex functions have nice properties related to minimization.

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Jointly convex functions

We say that a function $f(x, y)$ of two arguments $x \in \mathbb{R}^n, y \in \mathbb{R}^m$ is jointly convex if it is convex as a function of the variable $z = (x, y)$.
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**Theorem**

Let $f : \mathbb{R}^n \times \mathbb{R}^m \to (-\infty, \infty]$ be a jointly convex function. Then the function $g(x) = \inf_{y \in \mathbb{R}^m} f(x, y), \ x \in \mathbb{R}^n$, is convex.
Maxima of convex functions on compact convex sets

Theorem

Let $K \subset \mathbb{R}^N$ be a compact convex set, and $F : K \to \mathbb{R}$ be a convex function. Then $F$ attains its maximum at an extreme point of $K$. 
The convex conjugate

The convex conjugate is a very useful concept in convex analysis and optimization.
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Definition

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The convex conjugate \( F^* \) is always a convex function, no matter whether the function \( F \) is convex or not.
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The convex conjugate $F^*$ is always a convex function, no matter whether the function $F$ is convex or not. The definition of $F^*$ immediately gives the Fenchel (or Young, or Fenchel-Young) inequality

$$\langle x, y \rangle \leq F(x) + F^*(y) \quad \text{for all } x, y \in \mathbb{R}^N.$$  \hfill (45)
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1. The convex conjugate $F^*$ is lower semicontinuous.

2. The biconjugate $F^{**}$ is the largest lower semicontinuous convex function satisfying $F^{**}(x) \leq F(x)$ for all $x \in \mathbb{R}$. In particular, if $F$ is convex and lower semicontinuous then $F = F^{**}$. 
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5. *For \( z \in \mathbb{R}^N \) let \( F^{(z)} := F(x - z) \). Then \( (F^{(z)})^*(y) = \langle z, y \rangle + F^*(y) \).*
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The biconjugate \( F^{**} \) is sometimes also called the convex relaxation of \( F \) because of 2.
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Let us compute the convex conjugate for some examples.

Example

- Let \( F(x) = \frac{1}{2} \|x\|_2^2, \ x \in \mathbb{R}^N \). Then \( F^*(y) = \frac{1}{2} \|y\|_2^2 = F(y), \ y \in \mathbb{R}^N \). 

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we have

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For the converse inequality, we just set $x = y$ in the definition of the convex conjugate to obtain

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- Let $F(x) = \exp(x)$, $x \in \mathbb{R}$ - The map $x \mapsto xy - \exp(x)$ takes its maximum at $x = \ln y$ if $y > 0$ so that

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The subdifferential

The subdifferential generalizes the gradient for not necessarily differentiable functions.
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Definition
The subdifferential of a convex function $F: \mathbb{R}^N \to \mathbb{R}$ at a point $x \in \mathbb{R}^N$ is defined by

$$\partial F(x) = \{v \in \mathbb{R}^N : F(z) - F(x) \geq \langle v, z - x \rangle \text{ for all } z \in \mathbb{R}^N\}.$$  \hspace{1cm} (46)
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The subdifferential $\partial F(x)$ of a convex function $F$ is always non-empty. If $F$ is differentiable in $x$ then $\partial F(x)$ contains only the gradient,

$$\partial F(x) = \{\nabla F(x)\}.$$
A simple example of a function with a non-trivial subdifferential is the absolute value $F(x) = |x|$, for which

$$\partial F(x) = \begin{cases} \{\text{sign}(x)\} & \text{if } x \neq 0, \\ [-1, 1] & \text{if } x = 0, \end{cases}$$

where \text{sign}(x) = +1 for $x > 0$ and \text{sign}(x) = −1 for $x < 0$ as usual.
The subdifferential allows a simple characterization of minimizers of convex functions.

**Theorem**

A vector $\mathbf{x}$ is a minimum of $F$ if and only if $0 \in \partial F(\mathbf{x})$. 
The subdifferential and conjugation

Convex conjugate functions and subdifferentials are related in the following way.

**Theorem**

Let $F : \mathbb{R}^N \rightarrow (-\infty, \infty]$ be a convex function and $x, y \in \mathbb{R}^N$. 

1. $y \in \partial^* F(x)$ if and only if $x \in \partial F^*(y)$.

As a consequence, if $F$ is a convex lower semicontinuous function then $\partial F$ is the inverse of $\partial F^*$ in the sense that $y \in \partial^* F(x)$ if and only if $x \in \partial F^*(y)$. 
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Proximal mapping

Let \( F : \mathbb{R}^N \to (-\infty, \infty] \) be a convex function. Then, for \( z \in \mathbb{R}^N \) the function

\[
\mathbf{x} \mapsto F(\mathbf{x}) + \frac{1}{2} \| \mathbf{x} - \mathbf{z} \|_2^2
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is strictly convex due to the strict convexity of \( x \mapsto \| x \|_2^2 \).

By a previous result, its minimizer is unique.

The mapping

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P_F^z : \text{arg min} \ F(\mathbf{x}) + \frac{1}{2} \| \mathbf{x} - \mathbf{z} \|_2^2
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is called the proximal mapping associated with \( F \).

In the special case that \( F = \chi_K \) is the characteristic function of a convex set \( K \) defined in (44), then

\[
P_K^z = \text{arg min} \ \| \mathbf{x} - \mathbf{z} \|_2
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is the orthogonal projection onto \( K \), that is, \( P_K^z = \arg \min_{\mathbf{x} \in K} \| \mathbf{x} - \mathbf{z} \|_2 \).

If \( K \) is a subspace of \( \mathbb{R}^N \) then it is the usual orthogonal projection onto \( K \), and in particular, a linear map.
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Let $F : \mathbb{R}^N \rightarrow (-\infty, \infty]$ be a convex function. Then, for $z \in \mathbb{R}^N$ the function

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P_F(z) := \arg \min \left\{ F(x) + \frac{1}{2} \|x - z\|_2^2 : x \in \mathbb{R}^N \right\},
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Proposition

Let $F : \mathbb{R}^N \to (-\infty, \infty]$ be a convex function. Then $x = P_F(z)$ if and only if $z \in x + \partial F(x)$. 

The previous proposition justifies to write $P_F = I + \partial F$.

Moreau's identity relates the proximal mappings of $F$ and $F^{\ast}$.

Theorem (Moreau's Identity)

Let $F : \mathbb{R}^N \to (-\infty, \infty]$ be a lower semicontinuous convex function. Then, for all $z \in \mathbb{R}^N$, $P_F(z) = P_{F^{\ast}}(z)$.

If $P_F$ is easy to compute then the previous result shows that also $P_{F^{\ast}}(z) = z - P_F(z)$ is easy to compute.

It is useful to note that applying Moreau's identity to the function $\tau F$ for some $\tau > 0$ shows that $P_{\tau F}(z) = \tau P_{\tau F^{\ast}}(z)$.
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Proximal mapping

Proposition

Let \( F : \mathbb{R}^N \rightarrow (-\infty, \infty] \) be a convex function. Then \( x = P_F(z) \) if and only if \( z \in x + \partial F(x) \).

The previous proposition justifies to write

\[
P_F = (I + \partial F)^{-1}
\]

Moreau's identity relates the proximal mappings of \( F \) and \( F^* \).

Theorem (Moreau’s Identity)

Let \( F : \mathbb{R}^N \rightarrow (-\infty, \infty] \) be a lower semicontinuous convex function. Then, for all \( z \in \mathbb{R}^N \),

\[
P_F(z) + P_{F^*}(z) = z.
\]

If \( P_F \) is easy to compute then the previous result shows that also \( P_{F^*}(z) = z - P_F(z) \) is easy to compute. It is useful to note that applying Moreau’s identity to the function \( \tau F \) for some \( \tau > 0 \) shows that

\[
P_{\tau F}(z) + \tau P_{\tau^{-1}F^*}(z/\tau) = z. \tag{47}
\]
Nonexpansiveness of proximal mappings

Theorem
For a convex function $F : \mathbb{R}^N \rightarrow (-\infty, \infty]$ the proximal mapping $P_F$ is a contraction,

$$\|P_F(z) - P_F(z')\|_2 \leq \|z - z'\|_2 \quad \text{for all } z, z' \in \mathbb{R}^N.$$
Relevant example

Let $F(x) = |x|, x \in \mathbb{R}$, be the absolute value function.
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\[
P_{\tau F}(y) = \arg \min_{x \in \mathbb{R}} \left\{ \frac{1}{2} (x - y)^2 + \tau |x| \right\} = \begin{cases} y - \tau & \text{if } y \geq \tau, \\ 0 & \text{if } |y| \leq \tau, \\ y + \tau & \text{if } y \leq -\tau \end{cases}
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$= S_{\tau}(y)$.

(48)
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The function $S_\tau(y)$ is called soft thresholding or shrinkage operator.
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The function \( S_\tau(y) \) is called soft thresholding or shrinkage operator. More generally, if \( F(x) = \|x\|_1 \) is the \( \ell_1 \)-norm on \( \mathbb{R}^N \) then the minimization problem defining the proximal operator decouples and \( P_{\tau F}(y), y \in \mathbb{R}^N \) is given entrywise by

\[
P_{\tau F}(y)_l = S_\tau(y_l), \quad l \in [N].
\]

(49)
Optimization problems

An optimization problem is of the form

\[
\min_{x \in \mathbb{R}^N} F_0(x) \quad \text{subject to } Ax = y, \quad F_j(x) \leq b_j, \; j \in [M],
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(50) (51)
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$$\min_{x \in \mathbb{R}^N} F_0(x) \quad \text{subject to } Ax = y, \quad (50)$$

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where the function $F_0: \mathbb{R}^N \to (-\infty, \infty]$ is called objective function,
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\min_{\mathbf{x} \in \mathbb{R}^N} F_0(\mathbf{x}) \quad \text{subject to} \quad \mathbf{A}\mathbf{x} = \mathbf{y}, \quad (50)
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where the function \( F_0 : \mathbb{R}^N \rightarrow (-\infty, \infty] \) is called objective function, the functions \( F_1, ..., F_M : \mathbb{R}^N \rightarrow (-\infty, \infty] \) are called constraint functions,
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\begin{align*}
\min_{x \in \mathbb{R}^N} & \quad F_0(x) \quad \text{subject to} \quad Ax &= y, \\
& \quad F_j(x) \leq b_j, \quad j \in [M],
\end{align*}
\]

(50)

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$$\min_{\mathbf{x} \in \mathbb{R}^N} F_0(\mathbf{x}) \quad \text{subject to} \quad \mathbf{A}\mathbf{x} = \mathbf{y},$$

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We note that the equality constraint may be removed and represented by inequality constraints of the form

$$F_j(x) \leq y_j, \quad -F_j(x) \leq -y_j \quad \text{with} \quad F_j(x) = \langle A_j, x \rangle \quad \text{where} \quad A_j \in \mathbb{R}^N \quad \text{is a row of} \quad A.$$
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The set of feasible points described by the constraints is given by

$$K = \{ x \in \mathbb{R}^N : Ax = y, \quad F_j(x) \leq b_j, \quad j \in [M] \}.$$  \hspace{1cm} (52)
The optimization problem (50)-(51) is equivalent to the problem of minimizing $F_0$ over $K$,

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\min_{x \in K} F_0(x). \quad (53)
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Then our optimization problem becomes as well equivalent to the unconstrained optimization problem

$$\min_{x \in \mathbb{R}^N} F_0(x) + \chi_K(x).$$
Convex optimization problems

A convex optimization problem (or convex program) is a problem of the form (50)-(51), in which the objective function $F_0$ and the constraint functions $F_i$ are convex.
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Due to this equivalence we may freely switch between constrained and unconstrained optimization problems.
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Due to this equivalence we may freely switch between constrained and unconstrained optimization problems.

A *linear optimization problem* (or *linear program*) is one, where the objective function $F_0$ and all constraint functions $F_1, \ldots, F_M$ are linear. Clearly, this is a special case of a convex optimization problem.
Lagrange function

The *Lagrange function* of an optimization problem of the form (50)-(51) is defined for \( x \in \mathbb{R}^N, \xi \in \mathbb{R}^m, \nu \in \mathbb{R}^M, \nu_l \geq 0, \ l \in [M], \) by

\[
L(x, \xi, \nu) = F_0(x) + \xi^*(Ax - y) + \sum_{l=1}^{M} \nu_l(F_l(x) - b_l). \quad (54)
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For an optimization problem without inequality constraints we clearly set

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L(x, \xi) = F_0(x) + \xi^*(Ax - y). \tag{55}
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The *Lagrange dual function* is defined by

$$H(\xi, \nu) = \inf_{x \in \mathbb{R}^N} L(x, \xi, \nu), \quad \xi \in \mathbb{R}^m, \nu \in \mathbb{R}^M, \nu \geq 0.$$
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If \( x \mapsto L(x, \xi, \nu) \) is unbounded from below, then we set \( H(\xi, \nu) = -\infty \).
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\]

The dual function is always concave because it is the pointwise infimum of a family of affine functions, even if the original problem (50)-(51) is not convex.
Lagrange dual function

The dual function provides a bound on the optimal value of $F_0(x^\#)$ of the minimization problem (50),

$$H(\xi, \nu) \leq F(x^\#) \quad \text{for all } \xi \in \mathbb{R}^m, \nu \succeq 0.$$  \hfill (56)
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Indeed, if $x$ is a feasible point for (50) then $Ax - y = 0$ and $F_l(x) \leq b_l, \ l = 1, \ldots, M$, so that, for all $\xi \in \mathbb{R}^m$ and $\nu \geq 0$,

$$\xi^*(Ax - y) + \sum_{l=1}^{M} \nu_l(F_l(x) - b_l) \leq 0.$$
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Therefore,

$$L(x, \xi, \nu) = F_0(x) + \xi^*(Ax - y) + \sum_{l=1}^{M} \nu_l(F_l(x) - b_l) \leq F_0(x).$$
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Taking the infimum over all $x \in \mathbb{R}^N$ on the left hand side, and over all feasible $x$ on the right hand side shows (56).
Dual problem

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This optimization problem is called the dual problem to (50)-(51), which in this context is sometimes called primal problem. Since $H$ is concave this problem is equivalent to the convex optimization problem of minimizing the convex function $-H$ subject to the positivity constraint $\nu \geq 0$. 
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We would like this lower bound to be as tight as possible. This motivates to consider the optimization problem

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Weak and strong duality

Clearly

\[ H(\xi^#, \nu^#) \leq F(x^#). \]  \hfill (58)
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This inequality is called *weak duality*. For most (but not all) convex optimization problems even strong duality holds,

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\textbf{Theorem (Slater’s constraint qualification)}

\textit{Assume that} \( F_0, F_1, \ldots, F_M \) \textit{are convex functions with} \( \text{dom}(F_0) = \mathbb{R}^N \).
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In case that there are no inequality constraints, strong duality holds provided there exists $x$ with $Ax = y$, that is, if (50)-(51) is feasible.