Basis Pursuit and Null Space Property

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Overview

1. Recovery of sparse vectors
   - $l_1$ - Minimization
   - $l_q$, $l_p$ - Minimization

2. Stability

3. Robustness
   - $l_1$ - Minimization
   - $l_q$, $l_p$ - Minimization

4. Low-Rank Matrix Recovery
Problem Set

\[
(P_1) : \quad \min_{z \in \mathbb{C}^N} \|z\|_1 \\
\text{subject to } Az = y
\]
The Null Space Property

Definition

A matrix $A \in \mathbb{K}^{m \times N}$ is said to satisfy the null space property relative to a set $S \subset [N]$ if

$$\|v_S\|_1 < \|v_{\bar{S}}\|_1 \quad \forall v \in \ker A \setminus \{0\}$$

It is said to satisfy the null space property of order $s$ if it satisfies the null space property relative to any set $S \subset [N]$ with $\text{card}(S) \leq s$. 
Existence of a unique Solution

Theorem

Given a matrix $A \in \mathbb{K}^{m \times N}$, every vector $x \in \mathbb{K}^N$ supported on a set $S$ is the unique solution of $l_1$ Minimization with $Ax = y$ if and only if $A$ satisfies the null space property relative to $S$. 
Nonconvex Minimization

The number of nonzero entries of a vector $z \in \mathbb{K}^N$ is approximated by the $q$-th power of its $l_q$ - quasinorm,

$$\sum_{j=1}^{N} |z_j|^q \quad \xrightarrow{q \to 0} \quad \sum_{j=1}^{N} 1_{\{z_j \neq 0\}} = \|z\|_0.$$ 

New Problem:

$$\min_{z \in \mathbb{C}^N} \|z\|_q \quad \text{subject to } Az = y$$
Theorem

Given a matrix $A \in \mathbb{C}^{m \times N}$ and $0 < q \leq 1$, every $s$-sparse vector $x \in \mathbb{C}^N$ is the unique solution of $(P_q)$ with $y = Ax$ if and only if for any set $S \subset [N]$ with $\text{card}(S) \leq s$,

$$\|v_S\|_q < \|v_{\bar{S}}\|_q \quad \forall v \in \ker A \setminus \{0\}$$
Theorem

*Given a matrix* $A \in \mathbb{C}^{m \times N}$ *and* $0 < p < q \leq 1$ *, if every* $s$*-sparse vector* $x \in \mathbb{C}^N$ *is the unique solution of* $(P_q)$ *with* $y = Ax$ *, then every* $s$*-sparse vector* $x \in \mathbb{C}^N$ *is also the unique solution of* $(P_p)$ *with* $y = Ax$. 
In reality the vectors, we try to recover are only close to sparse vectors.

→ we try to recover a vector $x \in \mathbb{C}^N$ with an error controlled by its distance to s-sparse vectors.
Stable Null Space Property

Definition

A matrix $A \in \mathbb{K}^{m \times N}$ is said to satisfy the *stable null space property* with constant $0 < \rho < 1$ relative to a set $S \subset [N]$ if

$$\|v_S\|_1 < \rho \cdot \|v_{\bar{S}}\|_1 \quad \forall v \in \ker A \setminus \{0\}$$

It is said to satisfy the stable null space property of order $s$ if it satisfies the stable null space property relative to any set $S \subset [N]$ with $\text{card}(S) \leq s$. 
Approximation

Theorem

Suppose that a matrix \( A \in \mathbb{C}^{m \times N} \) satisfies the stable null space property of order \( s \) with constant \( 0 < \rho < 1 \). Then, for any \( x \in \mathbb{C}^N \), a solution \( x^* \) of \((P_1)\) with \( y=Ax \) approximates the vector \( x \) with \( l_1 \) error

\[
\|x - x^*\|_1 \leq \frac{2(1 + \rho)}{(1 - \rho)} \sigma_s(x)_1
\]
Approximation

Theorem

The matrix $A \in \mathbb{C}^{m \times N}$ satisfies the stable null space property with constant $0 < \rho < 1$ relative to $S$ if and only if

$$\|z - x\|_1 \leq \frac{1 + \rho}{1 - \rho} (\|z\|_1 - \|x\|_1 + 2\|x_S\|_1)$$

for all vectors $x, z \in \mathbb{C}^N$ with $Az = Ax$. 
Problem set

The measurement vector \( y \in \mathbb{C}^m \) is only an approximation vector of \( Ax \in \mathbb{C}^m \)

\[
\| Ax - y \| \leq \eta \quad \text{for some } \eta > 0
\]

New Problem set:

\[
\min_{z \in \mathbb{C}^N} \| z \|_1 \quad \text{subject to } \| Ax - y \| \leq \eta
\]
Robust Null Space Property

Definition

The matrix $A \in \mathbb{C}^{m \times N}$ is said to satisfy the robust null space property with constants $0 < \rho < 1$ and $\tau > 0$ relative to a set $S \subset [N]$ if

$$\|v_S\|_1 \leq \rho \|v_{\bar{S}}\|_1 + \tau \|Av\|_1 \quad \text{for all} \ v \in \mathbb{C}^N.$$
Approximation

**Theorem**

*Suppose that a matrix $A \in \mathbb{C}^{m \times N}$ satisfies the robust null space property of order $s$ with constants $0 < \rho < 1$ and $\tau > 0$. Then, for any $x \in \mathbb{C}^N$, a solution $x^*$ of $(P_1, \eta)$ with $y = Ax + e$ and $\|e\| \leq \eta$ approximates the vector $x$ with $l_1$-error

$$
\|x - x^*\|_1 \leq \frac{2(1 + \rho)}{(1 - \rho)} \sigma_s(x)_1 + \frac{4\tau}{1 - \rho} \eta.
$$

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Stronger Theorem

Theorem

The matrix $A \in \mathbb{C}^{m \times N}$ is said to satisfy the robust null space property with constants $0 < \rho < 1$ and $\tau > 0$ relative to a set $S$ if and only if

$$\|z - x\|_1 \leq \frac{1 + \rho}{1 - \rho} (\|z\|_1 - \|x\|_1 + 2\|x_{\bar{S}}\|_1) + \frac{2\tau}{1 - \rho} \|A(z - x)\|$$

for all vectors $x, z \in \mathbb{C}^N$. 
Definition

Given $q \geq 1$, the matrix $A \in \mathbb{C}^{m \times N}$ is said to satisfy the $l_q$-robust null space property of order $s$ with constants $0 < \rho < 1$ and $\tau > 0$ if, for any set $S \subset [N]$ with $\text{card}(S) \leq s$,

$$\|v_S\|_q \leq \frac{\rho}{s^{1-1/q}} \|v_{\bar{S}}\|_1 + \tau \|Av\| \quad \text{for all } v \in \mathbb{C}^N.$$ 

$$\|v_S\|_p \leq s^{1/p-1/q} \|v_S\|_q \quad \text{for } 1 \leq p \leq q.$$
**Theorem**

*Given* $1 \leq p \leq q$, *suppose that the matrix* $A \in \mathbb{C}^{m \times N}$ *satisfies the* $q$-*robust null space property of order* $s$ *with constants* $0 < \rho < 1$ *and* $\tau > 0$. *Then, for any* $z, x \in \mathbb{C}^N$, 

$$\|z - x\|_p \leq \frac{C}{s^{1-1/q}} (\|z\|_1 - \|x\|_1 + 2\sigma_s(x)_1) + Ds^{1/p - 1/q} \|A(z - x)\|,$$

*where* $C := \frac{(1+\rho)^2}{(1-\rho)}$ *and* $D := \frac{(3+\rho)\tau}{(1-\rho)}$. 

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### Problem Set

<table>
<thead>
<tr>
<th>Low-Rank Matrix</th>
<th>Sparse Vector</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\text{rank}(Z) \leq r$</td>
<td>$\text{supp}(z) \leq s$</td>
</tr>
<tr>
<td>$\min_{Z \in \mathbb{C}^{n_1 \times n_2}} \text{rank}(Z)$</td>
<td>$\min_{z \in \mathbb{C}^N} |z|_0$</td>
</tr>
<tr>
<td>$\min_{Z \in \mathbb{C}^{n_1 \times n_2}} |Z|_* \text{ subject to } A(Z) = y$</td>
<td>$\min_{z \in \mathbb{C}^N} |z|_1 \text{ subject to } Az = y$</td>
</tr>
<tr>
<td>$|Z|<em>* := \sum</em>{j=1}^n \sigma_j(Z), \ n := \min{n_1, n_2}$</td>
<td></td>
</tr>
</tbody>
</table>

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Theorem

Given a linear map $A$ from $\mathbb{C}^{n_1 \times n_2}$ to $\mathbb{C}^m$, every matrix $X \in \mathbb{C}^{n_1 \times n_2}$ of rank at most $r$ is the unique solution of the Low-Rank Recovery with $A(Z) = y$ if and only if, for all $M \in \ker A \setminus \{0\}$ with singular values $\sigma_1(M) \geq \ldots \geq \sigma_n(M) \geq 0$, $n := \min\{n_1, n_2\}$,

$$\sum_{j=1}^{r} \sigma_j(M) < \sum_{j=r+1}^{n} \sigma_j(M)$$