Problem Set 9

1. Prove the following statements.

(a) Let $G$ be an arbitrary abelian and let $U$ be an open subset of a topological space $X$. For a function $f : U \to G$, define

$$f \in G(U) \iff \forall a \in U \exists \text{ neighborhood } V \ni a \text{: } f|V = \text{constant}.$$ 

For $V \subseteq U$, set $\rho^U_V f := f|V$. Then $G$ defines a sheaf called the sheaf of locally constant function on $X$ with values in $G$.

(b) The set of all meromorphic functions on $\mathbb{C}$ defines a (pre)sheaf.

(c) Let $U \subseteq \mathbb{C}$ be open and

$$\mathcal{E}(U) := \{f : U \to \mathbb{C} : f \text{ is infinitely often real differentiable}\}.$$ 

For $V \subseteq U$, set $\rho^U_V f := f|V$. Then, $\mathcal{E}$ is a (pre)sheaf on $\mathbb{C}$.

(d) Let $U$ be an open subset of $\mathbb{C}$. The set

$$\mathcal{O}^*(U) := \left\{ f \in \mathcal{O}(U) : \frac{1}{f} \in \mathcal{O}(U) \right\}$$

is an abelian group. For $V \subseteq U$, let $\rho^U_V f := f|V$. Then, $\mathcal{O}^*$ is a sheaf on $\mathbb{C}$ called the sheaf of germs of nowhere vanishing holomorphic functions.

(e) Let $G$ be an abelian group and $x \in X$ a fixed point in a topological space $X$. Let $U$ be an open subset of $X$. Define

$$G_x(U) := \begin{cases} G, & \text{if } x \in U, \\ \{0\}, & \text{if } x \notin U, \end{cases}$$

and restriction mappings

$$\rho^U_V := \begin{cases} \{0\} \ni 0 \mapsto 0 \in \{0\}, & x \notin U \land x \notin V \\ G \ni g \mapsto 0 \in \{0\}, & x \in U \land x \notin V \\ \text{id}_G, & x \in U \land x \in V, \end{cases}$$

where the $V \subseteq U$ are open subsets of $X$. Then, $G_x$ is a sheaf on $X$ called the skyscraper sheaf on $X$.

2. Show that Example 12.2.(2) does not define a sheaf if $G$ contains at least two distinct elements and $X$ two disjoint open subsets.