Problem Set 7

1. Show that $D : \hat{\mathbb{C}} \to \mathbb{Z}$ is a divisor on the Riemann sphere iff the set $\text{supp } D := \{ z \in \hat{\mathbb{C}} : D(z) \neq 0 \}$ is finite.

2. Let $D \in \text{Div}(\hat{\mathbb{C}})$. The mapping $\text{deg} : \text{Div}(\hat{\mathbb{C}}) \to \mathbb{Z}$ defined by

$$\text{deg } D := \sum_{z \in \hat{\mathbb{C}}} D(z)$$

is called the degree of the divisors $D$. Let $f \in \mathcal{M}(\hat{\mathbb{C}})$. Then every principal divisor $\text{div } f$ on $\hat{\mathbb{C}}$ satisfies $\text{deg } \text{div } f = 0$.

3. Let $\gamma := \sum_{\nu=1}^{N} \alpha_{\nu} \gamma_{\nu}$ be a 1-chain on $\hat{\mathbb{C}}$ where each $\gamma_{\nu} : [0,1] \to \hat{\mathbb{C}}$ is a path. Show that the boundary operator $\partial : C^1 \to C^0,$

$$\partial \gamma = \sum_{\nu=1}^{N} \alpha_{\nu} (\gamma_{\nu}(1) - \gamma_{\nu}(0)),$$

can be interpreted as a divisor $\partial \gamma$.

The divisor $\partial \gamma$ is called boundary divisor of the 1-chain $\gamma$. Show that

(a) $\text{deg } \partial \gamma = 0$,

(b) and that for every divisor $D$ on $\hat{\mathbb{C}}$ with $\text{deg } D = 0$ there exists a 1-chain $\gamma$ such that $D = \partial \gamma$.

4. Let $G \subseteq \mathbb{C}$ and $U \subset G$ be open. For $D_1, D_2 \in \text{Div}(G)$ define $D_1 \leq D_2$ if $D_1(z) \leq D_2(z)$, $\forall z \in G$. Two divisors $D_1, D_2 \in \text{Div}(G)$ are called equivalent if the difference $D_1 - D_2 \in \text{HDiv}(G)$.

For $D \in \text{Div}(G)$, define the set $\mathcal{O}_D(U)$ as follows:

$$\mathcal{O}_D(U) := \{ f \in \mathcal{M}(U) : \text{ord}_z f \geq -D(z) \}.$$

Give an interpretation of the set $\mathcal{O}_D(U)$ in terms of the orders of the zeros and poles of its elements.

(a) Show that for $D := 0$, $\mathcal{O}_0(U) = \mathcal{O}(U)$ and that $\mathcal{O}_0(\hat{\mathbb{C}}) \cong \mathbb{C}$.

(b) Let $D_1, D_2 \in \text{Div}(G)$ with $D_1 \leq D_2$. Show that then $\mathcal{O}_{D_1}(U) \subseteq \mathcal{O}_{D_2}(U)$.

(c) Suppose $D_1, D_2 \in \text{Div}(G)$ are equivalent divisors on $G$. Show that then there exists an isomorphism between the sets $\mathcal{O}_{D_1}(U)$ and $\mathcal{O}_{D_2}(U)$. Exhibit this isomorphism.

(d) Show that if $D \in \text{Div}(\hat{\mathbb{C}})$ with $\text{deg } D < 0$ then $\mathcal{O}_D(\hat{\mathbb{C}}) = \{0\}$. 