Problem Set 6

1. Use the Weierstrass Product Theorem Man to derive the following representation of $\sin \pi z$ as an infinite product:

$$
\sin \pi z = \pi z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right)
$$

Hints: Use

$$
cot \pi z = \frac{1}{\pi} \left[\frac{1}{z} + \sum_{j=1}^{\infty} \frac{2z}{z^2 - j^2}\right].
$$

2. Set $z := 1/2$ in the above product representation of $\sin \pi z$ and show that this produces Wallis’s Product Formula:

$$
\sqrt{\frac{\pi}{2}} = \lim_{n \to \infty} \frac{2 \cdot 4 \cdot 6 \cdots 2n}{1 \cdot 3 \cdot 5 \cdots (2n - 1)} \frac{1}{\sqrt{2n+1}}.
$$

3. Prove: Let $U \subset \mathbb{C}$ be open and $f \in \mathcal{M}(U)$. Then there exist functions $g, h \in \mathcal{O}(U)$ so that $f = g/h$. In other words: The field generated by the holomorphic functions on $U$ is the field of meromorphic functions on $U$.

4. The Euler Gamma function $\Gamma : \mathbb{C} \setminus (-\mathbb{N}_0) \to \mathbb{C}$ can be defined as follows:

$$
\frac{1}{\Gamma(z)} = ze^{\gamma z} \prod_{k=1}^{\infty} \left(1 + \frac{z}{k}\right) e^{-z/k},
$$

where the Euler-Mascheroni constant $\gamma$ is chosen such that $\Gamma(1) = 1$. Use this product representation to show that

$$
\gamma = \lim_{k \to \infty} \left(\sum_{\ell=1}^{k} \frac{1}{\ell} - \log k\right),
$$

and determine an approximate value for $\gamma$.

5. Using the product representations of $\sin(\pi \cdot)$ and $\Gamma$, derive the following formula:

$$
\Gamma(z)\Gamma(1 - z) = \frac{\pi}{\sin \pi z}, \quad z \notin \mathbb{Z}.
$$