Problem Set 5

1. Determine whether the following infinite products converge.

(a) \( \prod_{n=2}^{\infty} \left( 1 + (-1)^{n} \frac{1}{n} \right) \):

(b) \( \prod_{n=2}^{\infty} \left( 1 + (-1)^{n} \frac{1}{\sqrt{n}} \right) \).

2. Prove the following lemmas:

(a) **Lemma 7.6:** Let \( U \subseteq \mathbb{C} \), let \( \{ f_{\nu} : U \rightarrow \mathbb{C} : \nu \in \mathbb{N} \} \) be a sequence of functions, and \( f : U \rightarrow \mathbb{C} \) a function on \( U \). Assume that for \( z \in U \) the sequence \( \{ f_{\nu}(z) \} \) converges uniformly to \( f(z) \). Further assume that there exists a constant \( K \in \mathbb{R} \) such that \( \text{Re} f(z) \leq K, \forall z \in U \). Then \( \{ \exp f_{\nu}(z) \} \) converges uniformly to \( \exp f(z) \).

(b) **Lemma 7.7:** Suppose \( U \subseteq \mathbb{C} \) is compact and \( \{ f_{\nu} : U \rightarrow \mathbb{C} : \nu \in \mathbb{N} \} \) a sequence of continuous functions with the property that \( \sum_{\nu \in \mathbb{N}} f_{\nu}(z) \) converges absolutely and uniformly for \( z \in U \). Then the infinite product

\[ f(z) := \prod_{\nu \in \mathbb{N}} (1 + f_{\nu}(z)) \]

converges absolutely and uniformly for all \( z \in U \).

3. Prove the following:

**Theorem 7.8:** Let \( G \subseteq \mathbb{C} \) be a region and \( \{ f_{\nu} : \nu \in \mathbb{N} \} \subset \mathcal{O}(G) \) a sequence of non-vanishing holomorphic Functions. If the infinite series \( \sum_{\nu \in \mathbb{N}} f_{\nu} \) converges absolutely and uniformly on all compact subsets of \( G \) then \( \prod_{\nu \in \mathbb{N}} (1 + f_{\nu}) \) converges in \( \mathcal{O}(G) \) to an \( f \in \mathcal{O}(G) \).

[Note: Convergence in \( \mathcal{O}(G) \) is convergence relative to the metric of \( C(G) \).]

4. **(Blaschke Products)** Let \( K := \{ z \in \mathbb{C} : |z| < 1 \} \) be the unit disk and \( \{ \alpha_{\nu} : \nu \in \mathbb{N} \} \) a sequence of non-zero complex numbers in \( K \). Assume that

\[ \sum_{\nu \in \mathbb{N}} (1 - |\alpha_{\nu}|). \]

converges. Show that the infinite product

\[ f(z) := \prod_{\nu \in \mathbb{N}} \frac{\alpha_{\nu} - z}{1 - \alpha_{\nu} \bar{z}} \frac{|\alpha_{\nu}|}{\alpha_{\nu}} \]

converges uniformly for all \( |z| \leq r < 1 \) and defines a holomorphic function \( f : K \rightarrow K \) which has \( \{ \alpha_{\nu} : \nu \in \mathbb{N} \} \) as its only set of zeros.