Problem Set 3

1. Let $\Omega_f$ be the set of periods of a function $f : \mathbb{C} \to \mathbb{C}$. Show that $\Omega_f$ is a subgroup of the additive group $(\mathbb{C}, +)$.

2. Let $\Omega_f$ be the set of periods of $f \in \mathcal{M}(\mathbb{C}) \setminus \mathbb{C}$. Show that $\Omega_f$ is a discrete subset of $\mathbb{C}$.

3. Let $G \subseteq \mathbb{C}$ be a region and $f \in \mathcal{O}(G) \setminus \mathbb{C}$. Prove that $f(G)$ is open in $\mathbb{C}$, i.e., $f$ is an open mapping.

4. Construct the Riemann surface for $f(z) := \sqrt{z^2 - 1}$.

5. Suppose $f$ is holomorph in a neighborhood of $z_0$ holomorph and not constant there. Moreover, suppose that $f$ takes the value $w_0$ at $z_0$ with multiplicity one. Show that there exist neighborhoods $U_\delta(z_0)$ and $U_\varepsilon(w_0)$ such that for every $w \in U_\varepsilon(w_0)$ there exists exactly one $z \in U_\delta(z_0)$ with $f(z) = w$. Furthermore, show that the mapping $f^{-1} : w \to z, \quad w \in U_\varepsilon(w_0)$, is holomorphic and $f^{-1}(w) = \frac{1}{2\pi i} \int_K \frac{\zeta f'(\zeta)}{f(\zeta) - w} d\zeta,$ where $K$ is a circle entirely contained in $U_\delta(z_0)$.

6. Show that a $\mathbb{R}$-linear mapping $L := Pdx + Qdy \in \mathcal{L}_\mathbb{R}(\mathbb{C})$ is $\mathbb{C}$-linear iff $L(iz) = iL(z)$ for all $z \in \mathbb{C}$. In other words, $L$ is $\mathbb{C}$-linear iff $Q = iP$.

7. Denote by $\mathcal{L}_\mathbb{C}(\mathbb{C}) \subseteq \mathcal{L}_\mathbb{R}(\mathbb{C})$ the vector space of all $\mathbb{C}$-linear mappings $L \in \mathcal{L}_\mathbb{R}(\mathbb{C})$ and $\overline{\mathcal{L}_\mathbb{C}(\mathbb{C})} \subseteq \mathcal{L}_\mathbb{R}(\mathbb{C})$ the vector space of all $\mathbb{C}$-antilinear mappings, i.e., $L(\alpha z) = \overline{\alpha}L(z)$, for all $\alpha, z \in \mathbb{C}$. Show that $\mathcal{L}_\mathbb{R}(\mathbb{C}) = \mathcal{L}_\mathbb{C}(\mathbb{C}) \oplus \overline{\mathcal{L}_\mathbb{C}(\mathbb{C})}$ and that $\{dz, d\overline{z}\}$ is a $\mathbb{C}$-basis for $\mathcal{L}_\mathbb{R}(\mathbb{C})$.

8. Let $L \in \mathcal{L}_\mathbb{R}(\mathbb{C})$. Express $L$ in the bases $\{dx, dy\}$ and $\{dz, d\overline{z}\}$ as $L = Pdx + Qdy = Adz + Bdz$ and exhibit relations between the coefficients $A, B$ and $P, Q$. Furthermore, show that $L \in \mathcal{L}_\mathbb{C}(\mathbb{C})$ iff $B = 0$ and that this condition is equivalent to the Cauchy Riemann equations.

9. Determine under what condition(s) the following identities hold:
   
   (a) $\log(z_1 z_2) = \log z_1 + \log z_2$ \quad ($z_1, z_2 \in \mathbb{C}$).
   
   (b) $\sqrt{z_1 z_2} = \sqrt{z_1} \sqrt{z_2}$ \quad ($z_1, z_2 \in \mathbb{C}$).