\[ f(0) = 0 \land |f'(0)| \geq 1 \notin \emptyset \]

Claim: \( A \) is compact (in \( \mathcal{O}(G) \)).

Proof. Boundedness is clear \( (f(G) \subseteq \overline{D}) \)

Let \( \{f_\nu\} \subseteq A \) with \( f_\nu \to f_\infty \) \( f_\nu \in \mathcal{O}(G) \)

Also \( f(0) = 0 \) and \( |f'(0)| = \lim_\nu |f'_\nu(0)| \geq 1 \).

As \( f \) is not constant \( f(G) \subseteq \overline{D} \) is open.

Thus \( f(G) \subseteq \overline{D} \). By Hurwitz, \( f \) is univalent,
and therefore \( f \in A \). \( \square \)

The mapping \( f \mapsto |f'(0)| \) is continuous and by the compactness of \( A \), \( \exists \, \hat{g} \in A : \)

\[ |\hat{g}'(0)| = \sup_{f \in A} |f'(0)| \geq 1. \]

Claim. \( \hat{g}(G) = \overline{D} \), \( \text{i.e.} \) \( \hat{g} \) is a biholomorphic mapping from \( G \) onto \( \overline{D} \).

Proof.

(a) Note: For \( a \in \overline{D} \), the mapping \( \varphi: \overline{D} \to \overline{D} \)

given by \( \varphi(z) = \frac{z-a}{\overline{z}-\overline{a}} \)

is a biholomorphic mapping with \( \varphi(a) = 0 \)
and \( \varphi^{-1} = \varphi \). (Exercise!)
(b) **Assumption**: \( \exists a \in D \setminus g(G) \).

Then \( a \neq 0 \) and \( \exists b 
eq 0 \) with \( a = b^2 \).

Define \( \varphi, \psi, p \in \Theta(D) \) and \( q \in \Theta(G) \) by

\[
\varphi(z) = \frac{z-a}{az-1}, \quad \psi(z) = \frac{z-b}{bz-1},
\]

\[
p(z) = \varphi(\psi(z)^2), \quad q(z) = \varphi(q(z))
\]

Then:

(i) \( p(0) = \varphi(\psi(0)^2) = \varphi(b^2) = \varphi(a) = 0 \)

\( p(D) \subseteq D \) and, since \( p \) is not univalent,

\[ |p'(0)| < 1 \quad \text{(by Schwarz's lemma)}. \]

(ii) \( q(0) = \varphi(q(0)) = \varphi(0) = a \).

\( q \) is univalent \( \Rightarrow \) \( q(G) \subseteq D \setminus \{0\} \)

(\( \text{since} \ a \notin q(G) \)). \( q \) is simply connected.

Therefore, \( \exists h \in \Theta(G) \) on \( q(G) \) with \( h^2 = q \) and \( h(0) = b \).

(See Lemma 9.3, applied to a simply connected region \( G \)).

I.e., \( h(G) \subseteq D \) and \( h \) is univalent (\( q \) is univalent!). Hence, \( f = \psi \circ h \in \Theta(G) \) is univalent and \( f(G) \subseteq D \), \( f(0) = \psi(h(0)) = \psi(b) = 0 \).

Using \( \varphi \circ \psi = \psi \circ \psi = \text{id}_D \), one gets

\[
p(f(z)) = \varphi(\psi(f(z))^2) = \varphi((\psi \circ \psi \circ h)(z)^2)
\]
\[ \varphi (h(z)^2) = \varphi (q(z)) = \varphi \circ \varphi (g(z)) = g(z) \]

Therefore, \( p'(0) \cdot f'(0) = g'(0) \)

\[ |f'(0)| = \frac{|g'(0)|}{|p'(0)|} > |g'(0)| > 1 \]

Hence \( f \in A \) if choice of \( g \) as the sup.

2. \( G \) is not dense in \( \mathbb{C} \)

Then \( \exists a \in \mathbb{C} \) and a \( \rho > 0 \) such that

\[ |z - a| > 2\rho \quad \forall z \in G. \]

Choose a \( b \in \mathbb{G} \) and define \( f \in \Theta(G) \) by

\[ f(z) = \frac{\rho}{z - a}. \]

Then \( |f(z)| < \frac{1}{\rho}, \forall z \in G. \) Hence,

\[ g(z) := f(z) - f(b) \]

is univalent, \( g(G) \subseteq D \), and \( g(b) = 0. \)

Therefore, \( G \) is biholomorphically equivalent to \( g(G) \) and, because of 1., also to \( D \).

3. General case. As \( G \neq \emptyset \), \( \exists a \in \mathbb{C} \setminus G, \)

which w.l.o.g. is assumed to be 0.

Then \( G \subseteq \mathbb{C} \setminus \{0\} \) and since \( G \) is simply connected,

\( \exists f \in \Theta(G) \) such that \( f(0)^2 = z \) (Lemma 9.1).

Also, \( f \) is univalent (unique branch of \( \sqrt{z} \)).
and \( H := f(G) \), \( f : G \to H \) is biholomorphic.

Suppose \( \exists c \in H \cap (-H) \), i.e., \( \exists u, v \in G \)
such that \( c = f(u) = -f(v) \).

But then
\[
\begin{align*}
u &= f(w)^2 = (-f(v))^2 = v \\
\text{and } f(w) &= -f(v) \implies f(w) = 0 \quad (1)
\end{align*}
\]

Hence
\[
-H \subseteq \mathbb{C}_H
\]
and \( H \) is not dense in \( \mathbb{C} \). Now apply 2.

12. Sheaves and Sheaf Cohomology

Motivation.

Mittag-Leffler: Let \( \{ U_i \}_{i \in I} \) (I index set) be
an open cover of \( X \) (= \( \mathbb{C} \) or \( \mathbb{C}^2 \)). Suppose on
each \( U_i \) \( \exists \) meromorphic \( f_i : U_i \to \mathbb{C} \).
Does there exist a (global) meromorphic \( f : X \to \mathbb{C} \)
such that \( f = f_i \) is holomorphic
on \( U_i \), \( \forall i \in I \)? Yes.

Def 12.1

Let \( (X, \mathcal{X}) \) be a top. space (\( \mathcal{X} \): system of all
open subsets of \( X \)). A pre-sheaf of abelian
groups on \( X \) is a pair \( (\mathcal{F}, \rho) \) consisting of
(a) a family of abelian groups \( \mathcal{F} = (\mathcal{F}(U) : U \in \mathcal{X}) \)
(b) a family of group homomorphisms

\[ \rho = \{ \rho^U_{V} : U, V \in \mathcal{X} \wedge V \subseteq U \} \]

such that

\[ \rho^U_U = \text{id}_{\mathcal{F}(U)} \quad \forall U \in \mathcal{X} \]

\[ \rho^U_W = \rho^V_W \circ \rho^U_V \quad \forall U, V, W \in \mathcal{X} \]

with \( W \subseteq V \subseteq U \).

The group homomorphisms \( \rho^U_V \) are called restriction homomorphisms.

**Notation**

- Instead of \((\mathcal{F}, \rho)\) one writes \( \mathcal{F} \).
- \( f|_V \Leftrightarrow \rho^U_V(f) \), \( f \in \mathcal{F}(U) \)
- \( \rho^U_W = \rho^V_W \circ \rho^U_V \Leftrightarrow f|_W = (f|_V)|_W \)
- \( \rho^U_U = \text{id}_{\mathcal{F}(U)} \Leftrightarrow f|_U = f \), \( f \in \mathcal{F}(U) \).

**Examples**

1. Let \( U \in \mathcal{X} \) be arbitrary.

   Let \( \mathcal{G}(U) := \{ f : U \to \mathbb{C} : f \text{ is continuous} \} \).

   For \( V \in \mathcal{X} \) with \( V \subseteq U \), let \( \rho^U_V \) be the usual restriction mapping. Then \( \mathcal{G} \) is a
pre-sheaf of abelian groups (of vector spaces).

2. Let $G_i$ be a fixed abelian group.
   Define
   $$ F(U) := \begin{cases} 
   G_i & U \neq \emptyset \\
   \{e\} & U = \emptyset 
   \end{cases} $$
   ($e$ : identity element of $G_i$) and
   $$ \rho^U_v = \begin{cases} 
   \text{id}_{G_i} & V \neq \emptyset \\
   e & V = \emptyset 
   \end{cases} $$

   Then $G_i$ is a pre-sheaf of abelian groups.

3. Let $\Omega := C$. For $U \subseteq \Omega$, let
   $$ \Theta(U) := \{ f : U \to \omega : f \text{ is holomorphic on } U \} $$
   and let $\rho^U_v(f) := f|_V$ (if $V \subseteq U$).

   Then $\Theta$ is a pre-sheaf of abelian groups (vector spaces).

**Def 12.3.** A pre-sheaf $F$ over $(\Omega, \tau)$ is called a **sheaf**, if $\forall U \in \tau$ and every family of subsets $\{ U_i \in \tau \}_{i \in I}$ with $U = \bigcup_{i \in I} U_i$ the following hold:

\begin{itemize}
  \item[(G1)] $f, g \in F(U)$, $f|_{U_i} = g|_{U_i}$, $\forall i \in I \Rightarrow f = g$.
\end{itemize}
(G2) Suppose \( f_i \in F(U_i) \) satisfies
\[
\frac{f_i}{U_i|U_j} = \frac{f_j}{U_i|U_j} \quad \forall ij \in I
\]
Then \( \exists f \in F(U) \) with \( \frac{f}{U_i} = f_i, \forall i \in I. \)

N.B. \((G1) \Rightarrow f \) if \((G2)\) is unique.

Examples.

1. \( G \): Sheaf of continuous fts over \( X \).
2. \( G \): not a sheaf
3. \( O \): Sheaf of holomorphic fts over \( \mathcal{O} \).

Let \((X, \mathcal{X})\) be a top. space and \( F \) a pre-sheaf of sets on \( X \) and \( x \in X \) arbitrary. Let \( U \in \mathcal{X} \).

The disjoint union
\[
|F| : = \bigcup_{x \in U} F(U)
\]
is called sheaf space.

Define \( \times \) on \(|F| \times |F|\) by
\[
\forall f \in F(U) \quad \forall g \in F(V) : 
\quad f \times g \iff \exists x \in W \subseteq U \cap V : f|_W = g|_W.
\]
\( \sim x \) is an equivalence relation.

Let \( F_x := |F|/\sim \) be the set of all equivalence relations.

**Def 12.5.** \( F_x \) is called the **stalk** of \( F \) at the point \( x \).