sum of its principal parts? Given a countably infinite numb of points in \( \mathbb{C} \), does there exist an \( f \in \mathcal{M}(\mathbb{C}) \) which has these points as its poles?

Let \( \mathcal{E} = \{ z_v : \nu = 1, \ldots , n \} \subseteq \mathbb{C} \) and let

\[
    h_v(z) = \sum_{\mathcal{E}} \frac{a_{\nu}}{(z - z_v)^{\mu}}
\]

be a finite principal part relative to \( \mathcal{E} \).

We define

\[
    f(z) = \sum_{\mathcal{E}} h_v(z) + g(z), \quad g \in \mathcal{O}(\mathbb{C})
\]

ci such a meromorphic function.

**Theorem (Mittag–Leffler)**

Let \( \{ z_v : \nu \in \mathbb{N}_0 \} \subseteq \mathbb{C} \) with \( 0 = |z_0| < \ldots < |z_n| \rightarrow \infty \).

Assume that to each \( z_v \) corresponds a finite principal part \( h_v \). Then there exists an \( f \in \mathcal{M}(\mathbb{C}) \) whose poles are given by \( \{ z_v : \nu \in \mathbb{N}_0 \} \) and whose principal parts are \( \{ h_v : \nu \in \mathbb{N}_0 \} \). Moreover, \( f \) is holomorphic on \( \mathcal{D}(\{ z_v : \nu \in \mathbb{N}_0 \}) \).

If \( f_1, f_2 \) are two such meromorphic functions then \( f_1 - f_2 = g \in \mathcal{O}(\mathbb{C}) \).

**Proof.** If \( z_0 = 0 \), discard \( f \) (for the moment).

Let \( \epsilon_v > 0, \nu \in \mathbb{N}_0 \), be such that \( \sum_{\nu=0}^{n} \epsilon_v \leq \infty \), choose a sequence \( \{ z_v : \nu \in \mathbb{N} \} \) with
\[0 < r_1 < \cdots < r_n \to \infty\]
\[r_n < \lambda, \quad \nu \in \mathbb{N}.
\]

For \(\nu \in \mathbb{N}_0\), let \(h_{\nu}(z) = \sum_{\mu=1}^{k_{\nu}} \frac{a_{\nu \mu}}{\mu! (z - \xi_\nu)^\mu}\) denote the principal parts.

All \(h_{\nu}(\nu \in \mathbb{N})\) are holomorphic in a neighborhood of \(z = 0\):
\[h_{\nu}(z) = \sum_{n=0}^{\infty} a_{\nu n} z^n, \quad a_{\nu n} \in \mathbb{C}, v \in \mathbb{N}.
\]

Furthermore, all these series converge uniformly in the disks \(K_0(\xi_{\nu}) = \{z \in \mathbb{C} : |z| < \xi_{\nu}\}\).

Now choose an \(m_{\nu} \in \mathbb{N}\) such that
\[g_{\nu}(z) = \sum_{n=0}^{m_{\nu}-1} a_{\nu n} z^n
\]
satisfies
\[|h_{\nu}(z) - g_{\nu}(z)| < \varepsilon_{\nu}, \quad \forall z \in K_0(\xi_{\nu}).
\]

As all \(g_{\nu}\) are polynomials, \(g_{\nu} \in \mathcal{O}(\mathbb{C})\).

Set \(g_0(z) \equiv 0\).

Claim: \(f : = \sum_{\nu=0}^{\infty} (h_{\nu} - g_{\nu})\) converges on \(\mathbb{C} \setminus \{\xi_{\nu} : \nu \in \mathbb{N}_0\}\).

It suffices to show that \(f\) converges in any disk
Choose \( N \in \mathbb{N} \) such that \( r_N > r \).

Then \( R_2 = \sum_{v=N}^{\infty} (h_v - g_v) \)

converges in all of \( K_0(r) \) and the fits \( h_v - g_v \) are holomorphic there. Thus \( R_2 \in \mathcal{O}(K_0(r)) \).

The series \( R_1 = \sum_{v=0}^{N-1} (h_v - g_v) \)

is holomorphic on \( K_0(r) \setminus \{ z_v : v \leq N \} \).

For \( z_k \in K_0(r) \), \( k < N \) and \( R_1 - h_k \)

is holomorphic in \( z_k \).

Hence \( f = R_1 + R_2 \)

has the required properties in \( K_0(r) \),

This holds for all \( r > 0 \), and the statement of

the theorem follows.

The second statement is clear.

9. The Weierstrass Product Theorem

Problem: Given finite collections \( \alpha = \{ \alpha_j : j=1, \ldots, m \} \subseteq \mathbb{C} \)

and \( \nu = \{ \nu_j : j=1, \ldots, m \} \subseteq \mathbb{N} \), then exists

a polynomial such that

\[
p(z) = \prod_{j=1}^{m} (z - \alpha_j)^{\nu_j}
\]

What is \( \alpha \) and \( \nu \) are infinite sets?
Def 9.1. Let $G \subset \mathbb{C}$ and $f : G \to \mathbb{C}$. The collection of all non-vanishing holomorphic fits on $G$ is denoted by $O^*(G)$.

Prop. 9.2. $(O^*(G), \cdot)$ is an abelian multiplicable group.

Proof. Exercise!

Lemma 9.3. Let $f \in O^*(G)$. Then $\exists g \in O(G)$ such that $f = e^g = \exp \circ g$.

Proof. $f \in O^*(G) \Rightarrow \frac{f'}{f} \in O(G)$.

i.e.,
\[
\frac{f'(z)}{f(z)} = \sum_{\nu=0}^{\infty} a_{\nu} z^\nu, \quad z \in G
\]

Let $g(z) = g(0) + \sum_{\nu=0}^{\infty} \frac{a_{\nu}}{\nu+1} z^{\nu+1}$, where $g(0)$ is yet to be determined.

Note: $g \in O(G)$ and $g' = \frac{f'}{f}$.

Need to show: $\frac{\exp(g)}{f} = 1$.

\[
\left(\frac{\exp(g)}{f}\right)' = g' \exp(g) f - \exp(g) f' = \frac{f^2}{f^2} \exp(g) \left[ g' f - f' \right] = 0.
\]
Thus \( \frac{\exp(g)}{f} \equiv c. \)

Now choose \( g(0) \) such that \( f(0) = \exp(g(0)) \).

Hence, \( \frac{\exp(g)}{f} \equiv 1. \)

\[ \text{N.B.} \text{ The most general solution to the above-mentioned problem for finite } \alpha \text{ and } \nu \text{ is of the form} \]

\[ p(z) \exp(g(z)) \]

where \( g \) is an arbitrary entire function.

**General case:**

Let \( \alpha := \{\alpha_j : j \in \mathbb{N}_0\} \subseteq \mathbb{C} \) and \( \nu := \{\nu_j : j \in \mathbb{N}_0\} \subseteq \mathbb{N}. \)

Find a holomorphic \( f \) whose only zeros are elements of \( \alpha \) and the multiplicities of the zeros are given by \( \nu \).

**Weierstrass – Product Theorem**

Let \( \alpha \) and \( \nu \) be defined as above but with \( 0 =: \alpha_0 \) and \( 0 < |\alpha_1| \leq \cdots \leq |\alpha_j| \leq \cdots \leq \infty. \)

Then \( \exists f_0 \in \mathcal{O}(\mathbb{C}) \) whose only zeros are given by \( \alpha \) and its multiplicities by \( \nu \). This \( f_0 \) is of
The form

\[ f_0(z) = z^{\nu_0} \prod_{j=1}^{\infty} \left( 1 - \frac{z}{\alpha_j} \right)^{\nu_j} \]

where the \( \alpha_j \)'s are polynomials given by the central segment of the power series

\[ -\log \left( 1 - \frac{z}{\alpha_j} \right) = \sum_{\ell=1}^{\infty} \frac{1}{\ell} \left( \frac{z}{\alpha_j} \right)^\ell \]

up to a certain term \( k_j \).

The most general fit is given by \( f_0 \exp(g_j) \) (where \( g_j \in \Theta(\varepsilon) \)).

**Proof.** Let \( f_1 \) be holomorphic in \( \alpha_j \) with \( f_1(\alpha_j) \neq 0 \).

Consider \( f(z) = (z - \alpha_j)^{\nu_j} f_1(z) \), \( z \neq \alpha_j \), \( j \in \mathbb{N} \).

\[ (\log f)'(z) = \frac{f_1(z)}{f(z)} = \frac{\nu_j}{z - \alpha_j} + \frac{f_1'(z)}{f_1(z)} \]

\( \text{pole of 1st order} \) with residue \( \nu_j \).

The sought-after fit \( f_0 \) has to have a zero at \( \alpha_j \) with multiplicity \( \nu_j \), but then the fit \( g_j := \frac{f_0}{f_0'} \) has a pole of order 1 with residue \( \nu_j \).

The corresponding principal part is: \( g_j(z) := \frac{\nu_j}{z - \alpha_j} \).
As in the proof of the Mittag-Leffler theorem,
\[
q_j(z) = - \nu_j \sum_{\mu=1}^{\infty} \frac{z^{\mu-1}}{\alpha_j^\mu}
\]
is approximated by the polynomial
\[
h_j(z) = - \nu_j \sum_{\mu=1}^{k_j} \frac{z^{\mu-1}}{\alpha_j^\mu}
\]
where \(k_j\) is chosen such that

\[
g(z) := \sum_{j=1}^{\infty} (q_j - h_j) \quad (9.1)
\]
converges uniformly on any bounded region.

Now let \(G \subseteq \mathbb{C}\) and suppose every cycle in \(\Gamma\) is null homologous in \(G\). Further suppose that \(x_0 = 0 \in G\) but all other \(x_j \notin G\). Let \(f(\bar{z}) \leq \delta\) be a path from 0 to \(z \in \mathbb{G}\).

By the Cauchy Integral Theorem

\[
F(z) := \int_{\gamma(z)} g(\xi) d\xi
\]
depends only on \(x\) but not \(f(\bar{z})\).

\(F \in \mathcal{O}(G)\) and since by (9.1) \(g\) converges uniformly
\(F\) can be differentiated \(k\) times yielding

\[
F(z) = \sum_{j=1}^{\infty} \nu_j \left[ \log \left(1 - \frac{z}{\alpha_j}\right) + \sum_{\mu=1}^{k_j} \frac{z^{\mu-1}}{\mu \alpha_j^\mu} \right]
\]
The fit \( f(z) = e^{-F(z)} = \prod_{j=1}^{\infty} \left( 1 - \frac{z}{\alpha_j} \right)^{\frac{k_j}{\mu_j}} e^{\left( \sum_{\mu=1}^{\infty} \frac{z^\mu}{\mu \omega_j^\mu} \right)^{\nu_j}} \) exists in \( O(G) \).

Let \( \varepsilon > 0 \). Then (proof of Mittag-Leffler.)

\( \exists j_0 \) such that \( \exp w_j \in O(G) \)

for all \( j > j_0(G) \) and \( \left| \frac{\alpha}{\sum_{j=j_0}^{\infty} w_j(z)} \right| < \varepsilon \)

for all \( j > j_0 \).

But, for \( 0 < \varepsilon < 1 \):

\[ |1 - \exp \left( \sum_{j=j_1}^{\infty} w_j(z) \right)| < 2 \varepsilon \]

I.e., \( \lim_{j_1 \to \infty} \exp \left( \sum_{j=j_1}^{\infty} w_j(z) \right) = 1 \).

Moreover,

\[ f(z) = \exp \left( F(z) \right) = \exp \left( \int_0^z g(\xi) \, d\xi \right) \]

\[ = \prod_{j=1}^{d_{l-1}} e^{w_j(z)} \cdot \exp \left( \sum_{j=j_1}^{\infty} w_j(z) \right) \]

= \prod_{j=1}^{\infty} e^{w_j(z)} \cdot \exp \left( \sum_{j=j_1}^{\infty} w_j(z) \right).

Now choose \( p_j(z) = \sum_{\mu=1}^{k_j} \frac{z^\mu}{\mu \omega_j^\mu} \). Then the fit

\[ f_0(z) = z^{\nu_0} \prod_{j=0}^{\infty} \left( 1 - \frac{z}{\alpha_j} \right)^{\nu_j} \exp \left( \nu_j p_j(z) \right) \]
satisfies all the requirements.

The second statement follows from Lemma 9.3