when \( F(\cdot, y_0) : E^1(U) \to \mathbb{C} \)

**Modern Cauchy:**

Let \( U \subseteq \mathbb{C} \) and simply connected (star shaped). Then every holomorphic Diff. form on \( U \) is null cohomologous.

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**7. Infinite Products**

Let \( \{c_v\}_{v \in \mathbb{N}_0} \subseteq \mathbb{C} \) be a sequence.

Let \( p_n = \prod_{v=0}^{\infty} (1 + c_v) \), \( n \in \mathbb{N}_0 \).

The sequence \( \{p_n\}_{n \in \mathbb{N}_0} \) is called an infinite product and is denoted by \( \prod_{v=0}^{\infty} (1 + c_v) \).

**Def 7.1.** If \( p = \lim_{n \to \infty} p_n \) exists and \( p \neq \{0, \infty\} \), then \( \prod_{v=0}^{\infty} (1 + c_v) \) is called convergent. If \( p_n \to 0 \) but no factor \( 1 + c_v \) is zero, or \( p = \infty \), then \( \prod_{v=0}^{\infty} (1 + c_v) \) is called divergent.

**N.B.** If a finite number of factors equals zero and if the removal of these factors yields a convergent product, then \( \prod_{v=0}^{\infty} (1 + c_v) = 0 \).

I.e., a convergent infinite product is zero if at least one factor is zero.
N.B. We will assume w.l.o.g. that all factors are nonzero.

As \( p_{n+1} = p_n (1 + c_{n+1}) = p_n + p_n c_{n+1} \), a necessary condition for convergence is

\[
\lim_{n \to \infty} c_n = 0.
\]

**Proposition 7.2.** \( \prod_{n=0}^{\infty} (1 + c_n) \) is convergent \( \iff \sum_{n=0}^{\infty} \log (1 + c_n) \) is convergent.

**Proof.** Assume w.l.o.g. that \( 1 + c_n \neq 0, \forall n \in \mathbb{N} \).

**Sufficiency:** Let \( s_n := \sum_{n=0}^{n} \log (1 + c_n) \).

Then \( \exp(s_n) = \prod_{n=0}^{n} (1 + c_n) \) and, as \( \exp \) is continuous, \( s_n \to s \), namely \( e^{s_n} \to e^{s} \).

**Necessity:** Assume \( \prod_{n=0}^{\infty} (1 + c_n) \) is convergent.

Then \( \forall \varepsilon > 0 \exists N_0 \in \mathbb{N} \forall n \geq N_0 : \forall n \in \mathbb{N} : \) \[
\left| \left( \frac{1 + c_{n_1}}{1 + c_{n_1 + 1}} \right) \left( \frac{1 + c_{n_1 + 1}}{1 + c_{n_1 + 2}} \right) \cdots \left( \frac{1 + c_{n_1 + m}}{1 + c_{n_1 + m + 1}} \right) - 1 \right| < \frac{\varepsilon}{2}.
\]

For \( \varepsilon \leq 1 \) : \( \log (1 + z) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1} z^k}{k} \).

If \( |z| < \frac{1}{2} \), then

\[
\left| \log (1 + z) \right| \leq \sum_{k=1}^{\infty} \frac{|z|^k}{k} \leq \sum_{k=1}^{\infty} |z|^k = \frac{|z|}{1 - |z|} \leq 2 |z|.
\]
For $|z| < \frac{1}{2} \varepsilon < \frac{1}{2}$:

\[
| \log (1 + z) | < \varepsilon.
\]

Now set $z = (1 - c_{u+1}) \cdots (1 - c_{u+m}) - 1$.

Then

\[
| \log (1 + c_{u+1}) \cdots (1 + c_{u+m}) | < \varepsilon,
\]

for $m > n_0$ and $m > 1$.

The multi-valuedness of $\log$ implies

\[
\log \prod_{v=1}^{u+m} (1 + cv) \neq \sum_{v=1}^{u+m} \log (1 + cv)
\]

\[
| \log \prod_{v=1}^{u+m} (1 + cv) | = \left| \sum_{v=u+1}^{u+m} \log (1 + cv) + 2\pi i q_m \right| < \varepsilon
\]

where $q_m \in \mathbb{Z}$.

Want to show that $q = 0$.

For $m > n_0$ and $m = 1$:

\[
| \log (1 + c_{u+1}) + 2\pi i q_1 | < \varepsilon
\]

Since $| \log (1 + c_{u+1}) | < \varepsilon \implies q_1 = 0$.

Thus

\[
| \log (1 + c_{u+1}) + \log (1 + c_{u+2}) + 2\pi i q_2 | < \varepsilon
\]

Hence, since $| \log (1 + c_{u+1}) | < \varepsilon \land | \log (1 + c_{u+2}) | < \varepsilon

\implies q_2 = 0$.

Proceeding along the same lines, yields $q_m = 0$. ♦
**Def 7.3.** \( \prod_{V=0}^{\infty} (1+c_v) \) is called **absolutely convergent** \( \iff \)
\[ \sum_{V=0}^{\infty} \log (1+c_v) \] is **absolutely convergent**.

**Theorem 7.4.** \( \prod_{V=0}^{\infty} (1+c_v) \) is abs. convergent \( \iff \)
\[ \sum_{V=0}^{\infty} c_v \] is abs. convergent.

**Proof.** A necessary condition for the convergence of \( \prod_{V=0}^{\infty} (1+c_v) \) is that
\[ \lim_{n \to \infty} c_n = 0. \]

Hence \( \exists n_0 \in \mathbb{N} \) \( \forall n > n_0 : |c_n| < \frac{1}{2} \).

Assume that \( c_n \neq 0, \forall n \in \mathbb{N} \).

\[
\left| 1 - \frac{\log (1+c_n)}{c_n} \right| = \left| \sum_{k=1}^{\infty} (-1)^k \frac{c_n^k}{k} \right|
\leq \frac{1}{2} \sum_{k=1}^{\infty} |c_n|^k = \frac{1}{2} \frac{|c_n|}{1-|c_n|} \leq \frac{1}{2}
\]

Thus,
\[ \frac{1}{2} |c_n| \leq |\log (1+c_n)| \leq \frac{3}{2} |c_n|. \] \((*)\)

(Note that these inequalities also hold for \( c_n = 0 \)!) Therefore,
\[ \sum_{V=0}^{\infty} \log (1+c_v) \] converges abs. \( \iff \)
\[ \sum_{V=0}^{\infty} |c_v| \] converges.
N.B. (*#) also holds for $c_{n} < ln(1)$. 

I.e., $\prod_{n=0}^{\infty} (1 + c_{n})$ converges abs. if $\sum_{n=0}^{\infty} (1 + l_{n})$ converges.

N.N.R. $\prod_{n=0}^{\infty} (1 + c_{n})$ can converge without $\sum_{n=0}^{\infty} c_{n}$ converging.

Example

Let $c_{2n-1} = n^{1/2}$ and $c_{2n} = -n^{1/2} + n^{-1}$, $n \in \mathbb{N}$.

Then $\sum_{n=1}^{\infty} c_{n} = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

But: $(1 + c_{2n-1})(1 + c_{2n}) = \cdots = 1 + n^{-3/2}$.

Hence $\prod_{n=1}^{\infty} (1 + c_{n}) = \prod_{n=1}^{\infty} (1 + n^{-3/2})$ converges,

because $\sum_{n=1}^{\infty} n^{-1/2}$ converges.

8. Theorem of Mittag-Leffler

Every rational function $R : \mathbb{C} \to \mathbb{C}$ can be written in the form

$$R(z) = \sum_{n=0}^{\infty} \frac{\sum_{\mu=1}^{k_{n}} a_{\mu} z^{\mu}}{(z - z_{\mu})^{\mu}} + \sum_{\eta=0}^{s} b_{\eta} z^{\eta},$$

where $a_{\mu}, b_{\eta} \in \mathbb{C}$.

Questions: Can every $f \in L_{1}(\mathbb{C})$ be represented as a
sum of its principal parts?
Given a countably infinite number of points in \( \mathcal{A} \), does
there exist an \( f \in \mathcal{M}(\mathcal{A}) \) which has these points
as its poles?

Let \( \mathcal{B} = \{ z_\nu : \nu = 1, \ldots, n \} \subseteq \mathcal{A} \) and let
\[
    \lambda \nu(z) := \sum_{\mu=1}^{k_\nu} \frac{a_{\nu \mu}}{(z - z_\nu)^{\mu}},
\]
be a finite principal part relative to \( \mathcal{B} \).

Rewrite \( f(z) = \sum_{\nu=1}^{\infty} \lambda \nu(z) + g(z) \), \( g \in \mathcal{O}(\mathcal{A}) \)
so as such a meromorphic \( f \).

**Theorem (Nevanlinna-Leffke)**

Let \( \{ z_{\nu} : \nu \in \mathbb{N}_0 \} \subseteq \mathcal{A} \) with \( 0 \leq |z_0| \leq \cdots \leq |z_{\nu}| \rightarrow \infty \)
Assume that to each \( z_{\nu} \) corresponds a finite principal
part \( \lambda_{\nu} \). Then there exists an \( f \in \mathcal{M}(\mathcal{A}) \) whose
poles are
given by \( \{ z_{\nu} : \nu \in \mathbb{N}_0 \} \) and whose principal parts are
\( \{ \lambda_{\nu} : \nu \in \mathbb{N}_0 \} \). Moreover, \( f \) is holomorphic on \( \mathcal{A}\setsminus \{ z_{\nu} : \nu \in \mathbb{N}_0 \} \).

If \( f, f_2 \) are two such meromorphic functions then
\( f_1 - f_2 = g \in \mathcal{O}(\mathcal{A}) \).

**Proof.** If \( z_0 = 0 \), discard \( f \) (for the moment).