Atomic characterizations of vector-valued function spaces

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Abstract

The first part of this diploma thesis deals with the topic of finding equivalent norms and characterizations for vector-valued Besov and Triebel-Lizorkin spaces $B^s_{p,q}(E)$ and $F^s_{p,q}(E)$. We will deduce general criteria by transferring and extending a theorem of Bui, Paluszyński and Taibleson from the scalar to the vector-valued case.

By using special norms and characterizations we will derive necessary and sufficient conditions for belonging to a vector-valued function spaces $B^s_{p,q}(E)$ or $F^s_{p,q}(E)$. It will be shown that an element of $\mathcal{S}'(\mathbb{R}^n, E)$ belongs to a function space if and only if it can be written as a linear combination of harmonic atoms resp. quarks with suitable conditions for the coefficients. Eventually, we will prove theorems 15.8 and 15.11 of [Tri97] dealing with atomic and subatomic representations.

Contents

Introduction

The aim of this work is to extend the results for atomic and subatomic charaterizations of the function spaces $B^s_{p,q}$ and $F^s_{p,q}$ to the vector-valued function spaces $B^s_{p,q}(E)$ and $F^s_{p,q}(E)$. For a comprehensive treatise of the scalar case ($E = \mathbb{C}$) we refer to chapter 13 and 14 of [Tri97]. A short consideration of the vector-valued case is given in chapter 15 of that book. But the proofs of the crucial theorems 15.8, p. 114 and 15.11, p. 116 are only shortly outlined and are mostly based on results for vector-valued function spaces which are well-known in the scalar, but have not yet been considered in the vector-valued case in detail.

This diploma thesis tries to derive these two theorems in wider detail, including the necessary steps before. In chapter 1 we will deal with the fundamentals of vector-valued functions and function spaces. We won't give any proofs mainly because most of them are similar to the scalar case. Many of these were treated in [Tri83].

In the second chapter we will prove a general result for equivalent norms and characterizations of vector-valued function spaces $B^s_{p,q}(E)$ and $F^s_{p,q}(E)$ in full detail. The scalar version ($E = \mathbb{C}$) of this theorem goes back to Bui, Paluszyński and Taibleson (see [BPT96] and [BPT97]), where the proof, which we will transfer to the vector-valued case, is given in this form in [Ryc99]. Nevertheless, there will be a little modification caused by some minor gap in the original proof. An earlier version with a bit worse, but more general conditions can be found in [Tri92], section 2.4, p. 100 for $F^s_{p,q}$ and section 2.5, p. 132 for $B^s_{p,q}$. In the following we use our result to obtain explicit norms and characterizations which we need to prove atomic and subatomic representations later on.

In the third chapter we will derive atomic and subatomic charaterizations for function spaces. We keep close to the approach suggested in [Tri97], theorem 15.8, p. 114. Thus we follow chapters 13 and 14 of [Tri97] and transfer the results to the vector-valued case, with minor modifications due to some imperfections in the original proof.

1 Mathematical fundamentals

In this chapter we will introduce notations, definitions and fundamental results used later on. This will be done without proofs. If necessary, we will refer to the relevant literature.

1.1 Vector-valued functions and distributions

1.1.1 Lebesgue spaces of vector-valued functions

Let E be a complex Banach space with norm $\|\cdot|E\|$ and let E' be its dual. With U_E we denote the set of all $x \in E$ with $\|x|E\| = 1$. Furthermore, let

$$B_r(x) := \{y \in E : \|x-y|E\| \le r\}, \quad B_r := B_r(0) \quad \text{and} \quad B = B_1.$$

Let $(M, \mathcal{M}, m)$ be a σ-finite measure space, which will be the space $\mathbb{R}^n$ with the σ-algebra of Borel sets and the Lebesgue measure $|\cdot|$ in the sequel. A function $f : M \to E$ is called E-measurable if there exists a subset M_0 of M such that $m(M_0) = 0$ and $f(M \setminus M_0)$ is contained in a separable subspace E_0 of E and if the complex-valued functions

$$a(f) : x \mapsto a(f(x))$$

are measurable for all $a \in E'$. In particular, every continous $f : \mathbb{R}^n \to E$ is measurable because its image is separable and the functions $a(f)$ are continous, too.

If f is E-measurable in this sense, then the function $\|f|E\| : M \to \mathbb{R}$, $x \mapsto \|f(x)|E\|$ is measurable because of

$$\|f(x)|E\| = \sup_{a \in U_{E'}} |a(f(x))|. \tag{1}$$

Therefore, we can define the spaces $L_p(E)$ for $0 < p \le \infty$ as follows:

$$L_p(M,E) := \left\{ f : M \to E, f \text{ measurable }, \Big\| \|f|E\| \Big| L_p(M, \mathcal{M}, m) \Big\| < \infty \right\}.$$

We write shortly $L_p(E) := L_p(\mathbb{R}^n, E)$ and $L_p := L_p(\mathbb{C})$. The spaces $L_p(M,E)$ are (quasi)-Banach spaces.

For functions $f : M \to E$ of the form

$$f = \sum_{k=1}^{K} b_k(x) u_k$$

with integrable $b_k : M \to \mathbb{C}$ and $u_k \in E$ for $k = 1, \dots, K$ we define the Bochner integral as a mapping into E through

$$\int_{\mathbb{R}^n} f(x)\, dx := \sum_{k=1}^{K} u_k \int_{\mathbb{R}^n} b_k(x)\, dx.$$

For every $a \in E'$ it follows

$$\begin{aligned} a\left(\int_{\mathbb{R}^n} f(x)\ dx\right) &= a\left(\sum_{k=1}^{K} u_k \int_{\mathbb{R}^n} b_k(x)\ dx\right) = \sum_{k=1}^{K} a(u_k)\int_{\mathbb{R}^n} b_k(x)\ dx \\ &= \int_{\mathbb{R}^n} \sum_{k=1}^{K} a(u_k) b_k(x)\ dx = \int_{\mathbb{R}^n} a(f(x))\ dx \end{aligned} \tag{2}$$

and thus with (1)

$$\left\| \int_{\mathbb{R}^n} f(x)\ dx \middle| E \right\| \leq \int_{\mathbb{R}^n} \|f(x)|E\|\ dx. \tag{3}$$

According to that the Bochner integral is a bounded linear operator from the subspace of functions of this form into E. This subspace is dense in $L_1(M, E)$ (see [Gra04], section 4.5.c., p. 318). So the operator can be continued to $L_1(M, E)$ uniquely. We want to call this continuation Bochner integral. Then the properties (2) and (3) hold for all $f \in L_1(M, E)$.

We define the Hardy-Littlewood maximal function $M(f)$ for $f \in L_1^{loc}$ as

$$M(f)(x) := \sup_{B_r(y) \ni x} \frac{1}{|B_r(y)|} \int_{B_r(y)} |f(y)|\ dy. \tag{4}$$

If, for a given $K : \mathbb{R}^n \to \mathbb{C}$, there exists a non-negative, monotonically decreasing function $\psi \in L_1((0,\infty))$ with $|K(x)| \leq \psi(|x|)$, then it holds

$$\sup_{\delta > 0} |K_\delta * f| \leq \|\psi(|\cdot|)|L_1(\mathbb{R}^n, \mathbb{C})\| \cdot M(f)(x) \tag{5}$$

for $K_\delta(x) := \delta^{-n} K(\delta^{-n} x)$ and $f \in L_1^{loc}$. A proof of this proposition can be found in [StW90], chapter 3, p. 59. Furthermore, for every $1 < p \leq \infty$ there exists a constant $c > 0$ such that

$$\|M(f)|L_p\| \leq c\|f|L_p\| \tag{6}$$

for all $f \in L_p$ and for every $1 < p < \infty$ and $1 < q \leq \infty$ there exists a constant $c > 0$ such that

$$\left\| \left(\sum_{j=1}^{\infty} M(f_j)^q \right)^{\frac{1}{q}} \middle| L_p \right\| \leq c \left\| \left(\sum_{j=1}^{\infty} |f_j|^q \right)^{\frac{1}{q}} \middle| L_p \right\| \tag{7}$$

for all $\{f_j\}_{j \in \mathbb{N}} \in L_p(l_q)$. The proofs go back to Hardy, Littlewood ($n = 1$) and Wiener ($n > 1$) resp. Fefferman and Stein (see [Tri92], section 2.2.2, p. 89 for references).

1.1.2 Vector-valued distributions

We denote by $\mathcal{S}(\mathbb{R}^n, E)$ the space of functions $\varphi : \mathbb{R}^n \to E$ which are infinitely often differentiable and for which the norms

$$\|\varphi|E\|_{K,L} := \sup_{x \in \mathbb{R}^n} (1 + |x|^2)^{\frac{K}{2}} \sum_{|\alpha| \leq L} \|D^\alpha \varphi(x)|E\|$$

for $K, L \in \mathbb{N}_0$ are finite. We write shortly $\mathcal{S}(\mathbb{R}^n) := \mathcal{S}(\mathbb{R}^n, \mathbb{C})$ and, for $\varphi \in \mathcal{S}(\mathbb{R}^n)$,

$$\|\varphi\|_{K,L} := \|\varphi|\mathbb{C}\|_{K,L}. \tag{8}$$

The Fourier transform $\hat{\varphi}$ of $\varphi \in \mathcal{S}(\mathbb{R}^n)$ will be defined as

$$\hat{\varphi}(\xi) := (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \varphi(x) e^{-ix\xi}\, dx,$$

whereas we denote the inverse Fourier transform by $\check{\varphi}$. It holds

$$\check{\varphi}(\xi) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \varphi(x) e^{ix\xi}\, dx.$$

We call a linear map $f : \mathcal{S}(\mathbb{R}^n) \to E$ an E-valued tempered distribution if there exist constants $c > 0$ and $K, L \in \mathbb{N}_0$ such that for all $\varphi \in \mathcal{S}(\mathbb{R}^n)$ we have

$$\|f(\varphi)|E\| \leq c\|\varphi\|_{K,L}.$$

The set of all this linear maps will be denoted by $\mathcal{S}'(\mathbb{R}^n, E)$. We say that f_j converges to f in $\mathcal{S}'(\mathbb{R}^n, E)$ if and only if $f_j(\varphi)$ converges to $f(\varphi)$ for all $\varphi \in \mathcal{S}(\mathbb{R}^n)$. Such a distribution f will be called regular if there is a measurable, locally Bochner integrable function $g : \mathbb{R}^n \to E$ so that

$$f(\varphi) = \int_{\mathbb{R}^n} g(x)\varphi(x)\, dx$$

for all $\varphi \in \mathcal{S}(\mathbb{R}^n)$. Such two functions g_1 and g_2 yield the same regular distribution if and only if they coincide almost everywhere. According to this $f \in L_p(E)$ for $1 \leq p \leq \infty$ can be unterstood as an element of $\mathcal{S}'(\mathbb{R}^n, E)$.

For an $f \in \mathcal{S}'(\mathbb{R}^n, E)$ we define the Fourier transform $\hat{f}$ as

$$\hat{f}(\varphi) := f(\hat{\varphi}) \text{ for } \varphi \in \mathcal{S}(\mathbb{R}^n).$$

The usual fundamental properties from the scalar case can be transfered. For instance if $f \in L_1(E)$, then $\hat{f} \in L_\infty(E)$ and it holds

$$\hat{f}(\xi) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} f(x) e^{-ix\xi}\, dx.$$

For $f \in \mathcal{S}'(\mathbb{R}^n, E)$ and $\psi \in \mathcal{S}(\mathbb{R}^n)$ we can define the convolution as

$$(\psi * f)(x) := (2\pi)^{-\frac{n}{2}} f(\psi(x - \cdot)) \text{ for } x \in \mathbb{R}^n, \tag{9}$$

analogously to the scalar case.[1] The function $\psi * f$ is infinitely often differentiable and there exist $c > 0$ and $K, L \in \mathbb{N}_0$ such that

$$\|(\psi * f)(x)|E\| \leq c(1 + |x|^2)^{\frac{K}{2}} \|\psi\|_{K,L}.$$

[1] Normally the constant $(2\pi)^{-\frac{n}{2}}$ is omitted in definition, but this form shortens later formulas a bit.

From the definition of the convolution we immediately get that if $f_j \in \mathcal{S}'(\mathbb{R}^n, E)$ converges to f, then $(\psi * f_j)(x)$ converges to $(\psi * f)(x)$ for all $x \in \mathbb{R}^n$. As in the scalar case the important relation

$$(\psi * f)\hat{} = \hat{\psi} \cdot \hat{f}.$$

holds. Furthermore, the convolution is associative, i.e. if $\psi, \varphi \in \mathcal{S}(\mathbb{R}^n)$ and $f \in \mathcal{S}'(\mathbb{R}^n, E)$, then

$$\varphi * (\psi * f) = (\varphi * \psi) * f.$$

1.2 Vector-valued function spaces

1.2.1 Lebesgue spaces of entire analytic functions

Let Ω be an open subset of $\mathbb{R}^n$. As in the scalar case we set

$$L_p^\Omega(\mathbb{R}^n, E) := \{f \in L_p(\mathbb{R}^n, E) \cap \mathcal{S}'(\mathbb{R}^n, E) : supp\ \hat{f} \subset \Omega\}$$

for $0 < p \le \infty$. Later on we will write shortly $L_p^\Omega(E)$ and L_p^Ω if $E = \mathbb{C}$. The Nikolskii inequality can be transfered to the vector-valued case, i.e. for $0 < p_1 < p_2 \le \infty$ there exists a constant $c > 0$ such that for all $r > 0$ and $f \in L_{p_1}^{B_r}(E)$ it holds

$$\|f|L_{p_2}(E)\| \le c r^{n\left(\frac{1}{p_1} - \frac{1}{p_2}\right)} \|f|L_{p_1}(E)\|. \tag{10}$$

For a proof see [ScS01], lemma 1, p. 6. But one can also reproduce the proof of the scalar case (see [Tri83], section 1.3.2., p. 18) easily. Moreover we have the usual relations for the convolution: Let $f \in L_p(E)$ for $1 \le p \le \infty$ and $g \in L_1$. Then

$$\|g * f|L_p(E)\| \le (2\pi)^{-\frac{n}{2}} \|g|L_1\| \cdot \|f|L_p(E)\|. \tag{11}$$

If otherwise $0 < p < 1$, then there exists a constant $c > 0$ such that for $f \in L_p^{B_r}(E)$, $g \in L_p^{B_r}$ and $r > 0$

$$\|g * f|L_p(E)\| \le c r^{n\left(\frac{1}{p} - 1\right)} \|g|L_p\| \cdot \|f|L_p(E)\|. \tag{12}$$

The proof, which is based on (10), can be found in [Tri83], section 1.5.1., p. 25. Additionally, one gets for $0 < p \le \infty$, $a > \frac{n}{p}$ and $f \in L_p^{B_r}(E)$

$$\left\| \sup_{z \in \mathbb{R}^n} \frac{\|f(\cdot - z)|E\|}{(1 + r|z|)^a} \Big| L_p \right\| \le c \|f|L_p(E)\|. \tag{13}$$

A proof for $E = \mathbb{C}$ is given in [Tri83], section 1.4.1., p. 22.

1.2.2 Definition and basic properties of vector-valued function spaces

Let φ_j for $j \in \mathbb{N}_0$ be elements of $\mathcal{S}(\mathbb{R}^n)$ with

$$\begin{aligned} &supp\ \varphi_0 \subset \{|\xi| \le 2\}, \\ &supp\ \varphi_j \subset \{2^{j-1} \le |\xi| \le 2^{j+1}\} \text{ for } j \in \mathbb{N}, \\ &\sum_{j=0}^{\infty} \varphi_j(\xi) = 1 \text{ for all } \xi \in \mathbb{R}^n, \\ &|D^\alpha \varphi_j(\xi)| \le c_\alpha 2^{-j|\alpha|} \text{ for all } \alpha \in \mathbb{N}_0^n. \end{aligned} \tag{14}$$

Then we call $\{\varphi_j\}_{j=0}^\infty$ a smooth dyadic resolution of unity. For instance one can choose $\Psi \in \mathcal{S}(\mathbb{R}^n)$ with $\Psi(\xi) = 1$ for $|\xi| \leq 1$ and *supp* $\Psi \subset \{|\xi| \leq 2\}$ and set

$$\varphi_0(\xi) := \Psi(\xi), \quad \varphi_1(\xi) := \Psi(\xi/2) - \Psi(\xi), \quad \varphi_j(\xi) := \varphi_1(2^{-j+1}\xi) \text{ für } j = 2, \dots$$

For a smooth dyadic resolution of unity $\{\varphi_j\}_{j=0}^\infty$ and $f \in \mathcal{S}'(\mathbb{R}^n, E)$ it holds

$$f = \sum_{j=0}^{\infty} (\varphi_j \hat{f})\check{}$$

in $\mathcal{S}'(\mathbb{R}^n, E)$.

Now we have collected the essentials for defining the vector-valued function spaces.

Definition 1.1. Let $0 < p \leq \infty$, $0 < q \leq \infty$, $s \in \mathbb{R}$ and $\{\varphi_j\}_{j=0}^\infty$ be a smooth dyadic resolution of unity. For $f \in \mathcal{S}'(\mathbb{R}^n, E)$ we define

$$\|f|B^s_{p,q}(E)\| := \left(\sum_{j=0}^{\infty} 2^{jsq} \|(\varphi_j \hat{f})\check{}\,|L_p(E)\|^q \right)^{\frac{1}{q}}$$

(modified if $q = \infty$) with

$$B^s_{p,q}(E) := \left\{ f \in \mathcal{S}'(\mathbb{R}^n, E) : \|f|B^s_{p,q}(E)\| < \infty \right\}.$$

Definition 1.2. Let $0 < p < \infty$, $0 < q \leq \infty$, $s \in \mathbb{R}$ und $\{\varphi_j\}_{j=0}^\infty$ be a smooth dyadic resolution of unity. For $f \in \mathcal{S}'(\mathbb{R}^n, E)$ we define

$$\|f|F^s_{p,q}(E)\| := \left\| \left(\sum_{j=0}^{\infty} 2^{jsq} \|(\varphi_j \hat{f})\check{}\,|E\|^q \right)^{\frac{1}{q}} \Big| L_p \right\|$$

(modified if $q = \infty$) with

$$F^s_{p,q}(E) := \left\{ f \in \mathcal{S}'(\mathbb{R}^n, E) : \|f|F^s_{p,q}(E)\| < \infty \right\}.$$

We will write shortly $B^s_{p,q}$ for $B^s_{p,q}(\mathbb{C})$ and $F^s_{p,q}$ for $F^s_{p,q}(\mathbb{C})$. As one can see, the definition depends on the choice of the smooth dyadic resolution of unity. But we can show as in the scalar case that the introduced quasi-norms[2] for two different smooth dyadic resolutions of unity are equivalent for fixed p, q and s, i. e. that the so defined spaces are equal. As in the scalar case this follows from Fourier multiplier theorems, see [Tri83], section 2.3.2., p. 46. Furthermore, the so defined spaces are (quasi)-Banach spaces. We have the fundamental embedding

$$B^s_{p,q_1}(E) \hookrightarrow B^s_{p,q_2}(E), \qquad F^s_{p,q_1}(E) \hookrightarrow F^s_{p,q_2}(E)$$

for $q_1 < q_2$ and

$$B^s_{p,\min(p,q)}(E) \hookrightarrow F^s_{p,q}(E) \hookrightarrow B^s_{p,\max(p,q)}(E).$$

[2]In the following we will use the term “norm“ even if we only have quasi-norms for $p < 1$ or $q < 1$.

Additionally we have for $1 < p < \infty$, as proven in [ScS01], theorem 1, p. 21,

$$B^0_{p,1}(E) \hookrightarrow F^0_{p,1}(E) \hookrightarrow L_p(E) \hookrightarrow F^0_{p,\infty}(E) \hookrightarrow B^0_{p,\infty}(E)$$

and

$$B^L_{\infty,1}(E) \hookrightarrow C^L_{ub}(E) \hookrightarrow B^L_{\infty,\infty}(E) \tag{15}$$

for all $L \in \mathbb{N}_0$ where $C^L_{ub}(E)$ is the set of all L times continuously differentiable functions $f : \mathbb{R}^n \to E$.

For $m : \mathbb{R}^n \to \mathbb{C}$ let

$$\|m\|_N := \sup_{|\alpha|\leq N} \sup_{x\in\mathbb{R}^n} (1 + |x|^2)^{\frac{|\alpha|}{2}} |D^\alpha m(x)|.$$

Then there exist $c > 0$ and $N \in \mathbb{N}$ in dependency of p, q and s such that for all infinitely often differentiable functions $m : \mathbb{R}^n \to \mathbb{C}$

$$\begin{aligned} \|(m\hat{f})\check{}\,|B^s_{p,q}(E)\| &\leq c\|m\|_N \cdot \|f|B^s_{p,q}(E)\| \text{ resp.} \\ \|(m\hat{f})\check{}\,|F^s_{p,q}(E)\| &\leq c\|m\|_N \cdot \|f|F^s_{p,q}(E)\|. \end{aligned} \tag{16}$$

This statement follows from Fourier multiplier theorems. For a proof in the scalar case see [Tri83], section 1.5.2., p. 26 und section 1.6.3., p. 31.

Let $I_\sigma(f) := ((1 + |\cdot|^2)^{\frac{\sigma}{2}} \hat{f})\check{}$. Then f is an element of $B^s_{p,q}(E)$ if and only if $I_\sigma(f)$ is an element of $B^{s-\sigma}_{p,q}(E)$ and we have

$$\|\cdot|B^s_{p,q}(E)\| \sim \|I_\sigma(\cdot)|B^{s-\sigma}_{p,q}(E)\|. \tag{17}$$

There is an obvious analogue for $F^s_{p,q}(E)$. A proof in the scalar case can be found in [Tri83], section 2.3.8., p. 58. But you can as well derive this proposition directly from the Fourier multiplier theorems [Tri83], section 1.5.2., p. 26 and section 1.6.3., p. 31, which also hold true in the vector-valued case.

For function spaces we have the so-called Sobolev embeddings: If $0 < p_0 \leq p_1 \leq \infty$, $0 < q \leq \infty$, then

$$B^{s_0}_{p_0,q} \hookrightarrow B^{s_1}_{p_1,q} \quad \text{if } s_0 - \frac{n}{p_0} = s_1 - \frac{n}{p_1}. \tag{18}$$

The proof relies on Nikolskii's inequality (see (10)). For a derivation (in the vector-valued case) see e. g. [ScS01], proposition 3, p. 12.

If $0 < p_0 < p_1 < \infty$, $0 < q_0, q_1 \leq \infty$, then

$$F^{s_0}_{p_0,q_0} \hookrightarrow F^{s_1}_{p_1,q_1} \quad \text{if } s_0 - \frac{n}{p_0} = s_1 - \frac{n}{p_1}. \tag{19}$$

The proof for the vector-valued case can be found in [ScS01], theorem 5, p. 36.

As in the scalar case we define $\mathscr{C}^s(E) := B^s_{\infty,\infty}$ and

$$\mathscr{C}^{-\infty}(E) := \bigcup_{s\in\mathbb{R}} \mathscr{C}^s(E).$$

By (18) and (19) we have

$$\mathscr{C}^{-\infty}(E) = \left\{ f \in \mathcal{S}'(\mathbb{R}^n, E) \text{ and } \exists\, p, q, s : f \in B^s_{p,q}(E) \vee f \in F^s_{p,q}(E) \right\}. \tag{20}$$

Furthermore, we set

$$\begin{aligned} \sigma_p &= n \left(\frac{1}{p} - 1 \right)_+ \\ \sigma_{p,q} &= n \left(\frac{1}{\min(p,q)} - 1 \right)_+ , \end{aligned} \tag{21}$$

where $a_+ = \max(a, 0)$. Let $\lfloor a \rfloor$ be the biggest integer smaller or equal to a and $\lceil a \rceil$ the smallest integer bigger or equal to a.

2 Equivalent norms and characterizations for vector-valued function spaces

In the first section of this chapter we will prove a theorem which gives equivalent norms and characterizations for function spaces $B^s_{p,q}(E)$ and $F^s_{p,q}(E)$ in a very general form. In view of notation we stay close to [Tri92] resp. [Tri97] here as well as in the later chapters such that some differences to the proof in [Ryc99], on which our derivations are based, cannot be avoided.

In the second part we apply the theorem to get explicit equivalent norms and characterizations which we will need later on for our characterization through atoms.

2.1 General characterizations

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a measurable function. We set $f_j(x) := 2^{jn} f(2^{jn}x)$.

Theorem 2.1. *Let $S + 1 \in \mathbb{N}_0$ with*

$$S \geq \lfloor s \rfloor, \tag{22}$$

let Ψ, $\psi \in \mathcal{S}(\mathbb{R}^n)$ and let there be an $\varepsilon > 0$ such that

$$|\Psi(x)| > 0 \text{ for } \{|x| < 2\varepsilon\}, \tag{23}$$

$$|\psi(x)| > 0 \text{ for } \left\{\frac{\varepsilon}{2} < |x| < 2\varepsilon\right\}, \tag{24}$$

$$D^\alpha \psi(0) = 0 \text{ for } |\alpha| \leq S. \tag{25}$$

Furthermore, let $s \in \mathbb{R}$ and

$$\begin{aligned} (\Psi^* f)_a(x) &:= \sup_{y \in \mathbb{R}^n} \frac{\|(\Psi \hat{f})\check{}(x-y)|E\|}{(1+|y|)^a} = \sup_{y \in \mathbb{R}^n} \frac{\|\left(\check{\Psi} * f\right)(x-y)|E\|}{(1+|y|)^a} \\ (\psi_j^* f)_a(x) &:= \sup_{y \in \mathbb{R}^n} \frac{\|(\psi(2^{-j}\cdot)\hat{f})\check{}(x-y)|E\|}{(1+2^j|y|)^a} = \sup_{y \in \mathbb{R}^n} \frac{\|\left(\check{\psi}_j * f\right)(x-y)|E\|}{(1+2^j|y|)^a}. \end{aligned} \tag{26}$$

(i) Let $0 < p \leq \infty$, $0 < q \leq \infty$ and $a > \frac{n}{p}$. Then

$$\|f|B^s_{p,q}(E)\|_{\Psi,\psi} := \|(\Psi\hat{f})\check{}|L_p(E)\| + \left(\sum_{j=1}^\infty 2^{jsq} \|(\psi(2^{-j}\cdot)\hat{f})\check{}|L_p(E)\|^q\right)^{\frac{1}{q}}$$

and

$$\|f|B^s_{p,q}(E)\|^a_{\Psi,\psi} := \|(\Psi^* f)_a|L_p\| + \left(\sum_{j=1}^\infty 2^{jsq} \|(\psi_j^* f)_a|L_p\|^q\right)^{\frac{1}{q}}$$

(modified in case of $q = \infty$) are equivalent norms for $\|\cdot|B^s_{p,q}(E)\|$. In addition, it holds

$$B^s_{p,q}(E) = \left\{f \in \mathcal{S}'(\mathbb{R}^n, E) : \|f|B^s_{p,q}(E)\|_{\Psi,\psi} < \infty\right\} \tag{27}$$

and

$$B^s_{p,q}(E) = \left\{ f \in \mathcal{S}'(\mathbb{R}^n, E) : \|f|B^s_{p,q}(E)\|^a_{\Psi,\psi} < \infty \right\}. \tag{28}$$

(ii) Let $0 < p < \infty$, $0 < q \le \infty$ *and* $a > \frac{n}{\min(p,q)}$. *Then*

$$\|f|F^s_{p,q}(E)\|_{\Psi,\psi} := \|(\Psi \hat{f})^\vee|L_p(E)\| + \left\| \left(\sum_{j=1}^{\infty} 2^{jsq} \|(\psi(2^{-j}\cdot)\hat{f})^\vee|E\|^q \right)^{\frac{1}{q}} |L_p \right\| \tag{29}$$

and

$$\|f|F^s_{p,q}(E)\|^a_{\Psi,\psi} := \|(\Psi^* f)_a|L_p\| + \left\| \left(\sum_{j=1}^{\infty} 2^{jsq} \left((\psi^*_j f)_a\right)^q \right)^{\frac{1}{q}} |L_p \right\| \tag{30}$$

(modified in case of $q = \infty$*) are equivalent norms for* $\|\cdot|F^s_{p,q}(E)\|$*. In addition, it holds*

$$F^s_{p,q}(E) = \left\{ f \in \mathcal{S}'(\mathbb{R}^n, E) : \|f|F^s_{p,q}(E)\|_{\Psi,\psi} < \infty \right\}$$

and

$$F^s_{p,q}(E) = \left\{ f \in \mathcal{S}'(\mathbb{R}^n, E) : \|f|F^s_{p,q}(E)\|^a_{\Psi,\psi} < \infty \right\}.$$

Proof. **First step:** Let Φ, $\varphi \in \mathcal{S}(\mathbb{R}^n)$ with

$$\begin{aligned} |\Phi(x)| &> 0 \text{ for } \{|x| < 2\varepsilon'\}, \\ |\varphi(x)| &> 0 \text{ for } \left\{\frac{\varepsilon'}{2} < |x| < 2\varepsilon'\right\} \end{aligned} \tag{31}$$

be given and let $(\Phi^* f)_a(x)$ and $(\varphi^*_j f)_a(x)$ be defined analogously as (26). Let $a > 0$, $0 < p \le \infty$ ($0 < p < \infty$ in case of $F^s_{p,q}(E)$), $0 < q \le \infty$ and $s < S + 1$ be fixed. We want to show in this step that there is a constant $C > 0$ independent of f such that

$$\begin{aligned} &\|(\Psi^* f)_a|L_p\| + \left(\sum_{j=1}^{\infty} 2^{jsq} \|(\psi^*_j f)_a|L_p\|^q \right)^{\frac{1}{q}} \\ &\qquad \le C\, \|(\Phi^* f)_a|L_p\| + C \left(\sum_{j=1}^{\infty} 2^{jsq} \|(\varphi^*_j f)_a|L_p\|^q \right)^{\frac{1}{q}} \end{aligned} \tag{32}$$

and

$$\begin{aligned} &\|(\Psi^* f)_a|L_p\| + \left\| \left(\sum_{j=1}^{\infty} 2^{jsq} \left((\psi^*_j f)_a\right)^q \right)^{\frac{1}{q}} |L_p \right\| \\ &\qquad \le C\, \|(\Phi^* f)_a|L_p\| + C \left\| \left(\sum_{j=1}^{\infty} 2^{jsq} (\varphi^*_j f)_a)^q \right)^{\frac{1}{q}} |L_p \right\| \end{aligned} \tag{33}$$

holds. We use the following:

Lemma 2.2. *Let Φ, $\varphi \in \mathcal{S}(\mathbb{R}^n)$ with (31) be given. Then there exist two functions $\Lambda, \lambda \in \mathcal{S}(\mathbb{R}^n)$ with*

$$\begin{aligned} supp\ \Lambda &\subset \{|x| < 2\varepsilon'\}\,, \\ supp\ \lambda &\subset \left\{\frac{\varepsilon'}{2} < |x| < 2\varepsilon'\right\}, \end{aligned} \tag{34}$$

$$\Lambda(x)\Phi(x)+\sum_{j=1}^{\infty}\lambda(2^{-j}x)\varphi(2^{-j}x) = 1. \tag{35}$$

Proof. Choose $\varrho \in \mathcal{S}(\mathbb{R}^n)$ such that $\varrho(x) > 0$ for $\left\{\frac{2}{3}\varepsilon' < |x| < \frac{5}{3}\varepsilon'\right\}$ and $supp\ \varrho \subset \left\{\frac{2\varepsilon'}{3} \leq |x| \leq \frac{5\varepsilon'}{3}\right\}$. Let

$$\zeta(x) := \begin{cases} \varrho(x)\frac{|\varphi(x)|}{\varphi(x)} & , \frac{2}{3}\varepsilon' \leq |x| \leq \frac{5}{3}\varepsilon' \\ 0 & , \text{ else} \end{cases},$$

which is infinitely often differentiable because of (31). Additionally, we have $supp\ \zeta \subset \left\{\frac{2}{3}\varepsilon' \leq |x| \leq \frac{5}{3}\varepsilon'\right\}$ and $\zeta(x)\varphi(x) > 0$ for $\left\{\frac{2}{3}\varepsilon' < |x| < \frac{5}{3}\varepsilon'\right\}$. If one sets

$$\lambda(x) := \begin{cases} \frac{\zeta(x)}{\zeta(\frac{x}{2})\varphi(\frac{x}{2})+\zeta(x)\varphi(x)+\zeta(2x)\varphi(2x)} & , \frac{1}{2}\varepsilon' \leq |x| \leq 2\varepsilon' \\ 0 & , \text{ else} \end{cases},$$

then λ is well-defined, $supp\ \lambda \subset \left\{\frac{1}{2}\varepsilon' < |x| < 2\varepsilon'\right\}$ and $\lambda \in \mathcal{S}(\mathbb{R}^n)$. Completing this by

$$\Lambda(x) := \begin{cases} \frac{1}{\Phi(x)}\left(1-\lambda(\frac{x}{2})\varphi(\frac{x}{2})\right) & , |x| \leq \frac{5}{3}\varepsilon' \\ 0 & , \text{ else} \end{cases},$$

we get (because of $\lambda(\frac{x}{2})\varphi(\frac{x}{2}) = 1$ on $\left\{\frac{5}{3}\varepsilon' \leq |x| \leq \frac{8}{3}\varepsilon'\right\}$ and (31)) that Λ is also well-defined, $supp\ \Lambda \subset \{|x| < 2\varepsilon'\}$ and $\Lambda \in \mathcal{S}(\mathbb{R}^n)$. For a fixed $j \geq 1$ and for all x with $2^j\varepsilon' \leq |x| < 2^{j+1}\varepsilon'$ we have

$$\begin{aligned} &\Lambda(x)\Phi(x) + \sum_{j=1}^{\infty}\lambda(2^{-j}x)\varphi(2^{-j}x) = \lambda(2^{-j-1}x)\varphi(2^{-j-1}x) + \lambda(2^{-j}x)\varphi(2^{-j}x) \\ &= \frac{\zeta(2^{-j-1}x)\varphi(2^{-j-1}x)}{\zeta(2^{-j-1}x)\varphi(2^{-j-1}x)+\zeta(2^{-j}x)\varphi(2^{-j}x)} + \frac{\zeta(2^{-j}x)\varphi(2^{-j}x)}{\zeta(2^{-j-1}x)\varphi(2^{-j-1}x)+\zeta(2^{-j}x)\varphi(2^{-j}x)} \\ &= 1 \end{aligned}$$

by the properties of the support of the function λ. For $|x| < 2\varepsilon'$ this results in

$$\begin{aligned} &\Lambda(x)\Phi(x) + \sum_{j=1}^{\infty}\lambda(2^{-j}x)\varphi(2^{-j}x) \\ &= \left[1-\lambda\left(\frac{x}{2}\right)\varphi\left(\frac{x}{2}\right)\right] + \lambda\left(\frac{x}{2}\right)\varphi\left(\frac{x}{2}\right) = 1. \end{aligned}$$

□

For our initial Φ, $\varphi \in \mathcal{S}(\mathbb{R}^n)$ we choose Λ, $\lambda \in \mathcal{S}(\mathbb{R}^n)$ by lemma 2.2. Because of the properties (34) and (35) we have the relation

$$\hat{f} = \Lambda(\cdot)\Phi(\cdot)\hat{f} + \sum_{k=1}^{\infty} \lambda(2^{-k}\cdot)\varphi(2^{-k}\cdot)\hat{f}$$

for all $f \in \mathcal{S}'(\mathbb{R}^n, E)$, convergence in $\mathcal{S}'(\mathbb{R}^n, E)$. Applying the Fourier transform and using properties of the convolution of functions from $\mathcal{S}(\mathbb{R}^n)$ with elements of $\mathcal{S}'(\mathbb{R}^n, E)$ (see (9)) yields

$$f = \left(\check{\Lambda} * \check{\Phi}\right) * f + \sum_{k=1}^{\infty} \left(\check{\lambda}_k * \check{\varphi}_k\right) * f$$

in $\mathcal{S}'(\mathbb{R}^n, E)$. Hence we can derive

$$(\check{\psi}_j * f)(y) = \left((\check{\psi}_j * \check{\Lambda}) * (\check{\Phi} * f)\right)(y) + \sum_{k=1}^{\infty} \left((\check{\psi}_j * \check{\lambda}_k) * (\check{\varphi}_k * f)\right)(y) \tag{36}$$

for all $y \in \mathbb{R}^n$. With the norm inequality of the Bochner integral (see (3)) it follows

$$\begin{aligned} \| \left(\left(\check{\psi}_j * \check{\lambda}_k\right) * (\check{\varphi}_k * f)\right)(y)|E\| &\leq \int_{\mathbb{R}^n} \left|\left(\check{\psi}_j * \check{\lambda}_k\right)(z)\right| \cdot \| \left(\check{\varphi}_k * f\right)(y-z)|E\| \, dz \\ &\leq (\varphi_k^* f)_a(y) \int_{\mathbb{R}^n} \left|\left(\check{\psi}_j * \check{\lambda}_k\right)(z)\right| (1 + 2^k|z|)^a \, dz \\ &\equiv (\varphi_k^* f)_a(y) \cdot I_{j,k}. \end{aligned} \tag{37}$$

The scalar(!) integral $I_{j,k}$ can be treated as in [Ryc99]:

Lemma 2.3. *Let $\mu, \nu \in \mathcal{S}(\mathbb{R}^n)$, $M \in \mathbb{Z}, M \geq -1$, $d > 0$ and*

$$D^\alpha \mu(0) = 0 \text{ for all } \alpha \in \mathbb{N}^n \text{ with } |\alpha| \leq M. \tag{38}$$

Then for all $N \in \mathbb{N}$ there exists a constant C_N such that for all $t \in (0, d]$

$$\sup_{z \in \mathbb{R}^n} |\left(\mu(t\cdot)^\vee * \check{\nu}\right)(z)| \, (1 + |z|)^N \leq C_N t^{M+1}.$$

Proof. By basic properties of the Fourier transform we have

$$\sup_{z \in \mathbb{R}^n} |\left(\mu(t\cdot)^\vee * \check{\nu}\right)(z)| \, (1 + |z|)^N \leq c_N \max_{|\alpha| \leq N+1} \|D^\alpha \left[(\mu(t\cdot)^\vee * \check{\nu})^\wedge\right] |L_1\|. \tag{39}$$

Using the Leibniz formula we obtain

$$\begin{aligned} |D^\alpha \left[(\mu(t\cdot)^\vee * \check{\nu})^\wedge\right](\xi)| &= |D^\alpha \left(\mu(t\cdot)\nu\right)(\xi)| \\ &\leq c_N \sum_{\beta \leq \alpha} t^{|\beta|} |(D^\beta \mu)(t\xi)(D^{\alpha-\beta}\nu)(\xi)|. \end{aligned}$$

From (38) and because of $\mu \in \mathcal{S}(\mathbb{R}^n)$ it follows

$$|(D^\beta \mu)(t\xi)| \leq c_{\mu,\beta} (t\xi)^{M-|\beta|+1} \text{ for } |\beta| \leq M.$$

If we use this and $|D^\beta \mu| \leq c_{\mu,\beta}$ for $\beta \in \mathbb{N}^n$ with $|\beta| > M$, we obtain

$$\sum_{\beta \leq \alpha} t^{|\beta|} |(D^\beta \mu)(t\xi)(D^{\alpha-\beta}\nu)(\xi)| \leq c_{\alpha,\mu} t^{M+1} \left(1 + |\xi|^{M+1}\right) \sum_{\beta \leq \alpha} \left|(D^{\alpha-\beta}\nu)(\xi)\right|.$$

We insert this result into (39) and get the desired result using $\nu \in \mathcal{S}(\mathbb{R}^n)$ where the constant depends on μ, ν and N. □

For $k \leq j$ we obtain

$$\begin{aligned}\left(\check{\psi}_j * \check{\lambda}_k\right)(z) &= \int_{\mathbb{R}^n} 2^{(k+j)n} \check{\psi}(2^j y)\ \check{\lambda}(2^k(z-y))\ dy \\ &= \int_{\mathbb{R}^n} 2^{jn} \check{\psi}(2^{j-k} y)\ \check{\lambda}(2^k z - y)\ dy \\ &= 2^{kn} \left(\check{\psi}_{j-k} * \check{\lambda}\right)(2^k z)\end{aligned}$$

by the substitution of variables $2^k y \to y$ and therefore

$$\begin{aligned}\int_{\mathbb{R}^n} \left|\left(\check{\psi}_j * \check{\lambda}_k\right)(z)\right| \cdot (1 + 2^k|z|)^a\ dz &= \int_{\mathbb{R}^n} 2^{kn} \left|\left(\check{\psi}_{j-k} * \check{\lambda}\right)(2^k z)\right| \cdot (1 + 2^k|z|)^a\ dz \\ &= \int_{\mathbb{R}^n} \left|\left(\check{\psi}_{j-k} * \check{\lambda}\right)(z)\right| \cdot (1 + |z|)^a\ dz \\ &\leq C_{\psi,\lambda} \sup_{z \in \mathbb{R}^n} \left|\left(\psi(2^{k-j}\cdot)^{\vee} * \check{\lambda}\right)(z)\right| \cdot (1 + |z|)^{a+n+1} \\ &\leq C'_{\psi,\lambda} 2^{(k-j)(S+1)}\end{aligned}$$

by lemma 2.3 with $\mu = \psi$ and $\nu = \lambda$ for $M = S$. In case of $k \geq j$ we deduce

$$\begin{aligned}\int_{\mathbb{R}^n} \left|\left(\check{\psi}_j * \check{\lambda}_k\right)(z)\right| \cdot (1 + 2^k|z|)^a\ dz &= \int_{\mathbb{R}^n} 2^{jn} \left|\left(\check{\psi} * \check{\lambda}_{k-j}\right)(2^j z)\right| \cdot (1 + 2^k|z|)^a\ dz \\ &= \int_{\mathbb{R}^n} \left|\left(\check{\psi} * \check{\lambda}_{k-j}\right)(z)\right| \cdot (1 + 2^{k-j}|z|)^a\ dz \\ &\leq 2^{(k-j)a} \int_{\mathbb{R}^n} \left|\left(\check{\psi} * \check{\lambda}_{k-j}\right)(z)\right| \cdot (1 + |z|)^a\ dz \\ &\leq C_{M,\psi,\lambda} 2^{(k-j)a} 2^{(j-k)(M+1)},\end{aligned}$$

where M can be chosen arbitrarily large since $(D^\alpha \lambda)(0) = 0$ for all $\alpha \in \mathbb{N}^n$ because of the properties of the support of λ (see (34)). If we choose $M \geq 2a - s$, we obtain the estimation

$$I_{j,k} \leq C_{\lambda,\psi} \begin{cases} 2^{(k-j)(S+1)} & , k \leq j \\ 2^{(j-k)(a-s+1)} & , k \geq j \end{cases}. \tag{40}$$

Furthermore, by definition of the maximal functions in (26)

$$\begin{aligned}(\varphi_k^* f)_a(y) &\leq (\varphi_k^* f)_a(x)(1 + 2^k|x-y|)^a \\ &\leq (\varphi_k^* f)_a(x) \max\left(1, 2^{(k-j)a}\right)(1 + 2^j|x-y|)^a.\end{aligned}$$

If we use this and insert it into (37) while applying (40), we get

$$\sup_{y\in\mathbb{R}^n} \frac{\| \left(\check{\psi}_j * \check{\lambda}_k * \check{\varphi}_k * f\right)(y)|E\|}{(1+2^j|x-y|)^a} \leq C_{\psi,\lambda}(\varphi_k^* f)_a(x) \left\{ \begin{array}{ll} 2^{(k-j)(S+1)} & ,k \leq j \\ 2^{(j-k)(-s+1)} & ,k \geq j \end{array} \right. . \tag{41}$$

In correspondence, if we replace λ_1 by Λ and φ_1 by Φ in the previous calculations, we obtain

$$\sup_{y\in\mathbb{R}^n} \frac{\| \left(\check{\psi}_j * \check{\Lambda} * \check{\Phi} * f\right)(y)|E\|}{(1+2^j|x-y|)^a} \leq C_{\Psi,\Lambda}(\Phi^* f)_a(x) 2^{-j(S+1)}. \tag{42}$$

One has to keep in mind that only the case $1 = k \leq j$ is needed, where we haven't used any conditions of the form $(D^\alpha \Lambda)(0) = 0$. With the representation of $\check{\psi}_j * f$ in (36) and with the triangle inequality for $\| \cdot |E\|$ we conclude

$$(\psi_j^* f)_a(x) \leq C(\Phi^* f)_a(x)\ 2^{-j(S+1)} + C\sum_{k=1}^{\infty}(\varphi_k^* f)_a(x) \left\{ \begin{array}{ll} 2^{(k-j)(S+1)} & ,k \leq j \\ 2^{(j-k)(-s+1)} & ,k \geq j \end{array} \right. .$$

By taking $\delta = \min(S+1-s, 1) > 0$ (by (22)) we arrive at

$$2^{js}(\psi_j^* f)_a(x) \leq C 2^{-j\delta}(\Phi^* f)_a(x) + C\sum_{k=1}^{\infty} 2^{ks}(\varphi_k^* f)_a(x) 2^{-|j-k|\delta}. \tag{43}$$

Analogously, by replacing ψ_1 by Ψ in the prior remarks, where we only used the case $k \geq j = 1$ and therefore conditions of the form $(D^\alpha \Psi)(0) = 0$ are not necessary, we get

$$(\Psi^* f)_a(x) \leq C(\Phi^* f)_a(x) + C\sum_{k=1}^{\infty} 2^{ks}(\varphi_k^* f)_a(x) 2^{-k\delta}. \tag{44}$$

Starting from this pointwise estimates we can now establish our assertions (32) and (33). For this we choose a usual method which is applied in [Tri92] several times (without further mentioning) and which is made into a lemma in [Ryc99]. We want to describe it in form of a lemma so that we can refer to this section later on.

Lemma 2.4. *Let $0 < p, q \leq \infty$ and $\delta > 0$. We assume that for the sequences of $\mathbb{R}$-measurable functions $\{g_k\}_{k=0}^\infty$ and $\{G_j\}_{j=0}^\infty$ it holds*

$$|G_j(x)| \leq C_0 \sum_{k=0}^{\infty} 2^{-|k-j|\delta}|g_k(x)| \ \text{for}\ \ x \in \mathbb{R}^n,$$

where C_0 is a constant independent of j and x. Then there exist constants C_1 and C_2 (in dependency of p, q, δ) such that

$$\left(\sum_{j=0}^{\infty} \|G_j|L_p\|^q\right)^{\frac{1}{q}} \leq C_1 \left(\sum_{j=0}^{\infty} \|g_j|L_p\|^q\right)^{\frac{1}{q}}, \tag{45}$$

$$\left\| \left(\sum_{j=0}^{\infty} |G_j|^q\right)^{\frac{1}{q}} |L_p \right\| \leq C_2 \left\| \left(\sum_{j=0}^{\infty} |g_j|^q\right)^{\frac{1}{q}} |L_p \right\|. \tag{46}$$

Proof. By Hölder's inequality for $1 \le q < \infty$ we have

$$\left(\sum_{j=0}^{\infty} |G_j(x)|^q\right)^{\frac{1}{q}} \le C_0 \left(\sum_{j=0}^{\infty} \left(\sum_{k=0}^{\infty} 2^{-|k-j|\left(\frac{1}{q}+\frac{1}{q'}\right)\delta} |g_k(x)|\right)^q\right)^{\frac{1}{q}}$$

$$\le C_0 \left(\sum_{j=0}^{\infty} \left(\sum_{k=0}^{\infty} 2^{-|k-j|\delta}\right)^{\frac{q}{q'}} \left(\sum_{k=0}^{\infty} 2^{-|k-j|\delta} |g_k(x)|^q\right)\right)^{\frac{1}{q}}$$

$$= C \left(\sum_{k=0}^{\infty} |g_k(x)|^q \left(\sum_{j=0}^{\infty} 2^{-|k-j|\delta}\right)\right)^{\frac{1}{q}}$$

$$= C_2 \left(\sum_{k=0}^{\infty} |g_k(x)|^q\right)^{\frac{1}{q}}.$$

In case of $0 < q < 1$ it holds

$$C_0 \left(\sum_{j=0}^{\infty} \left(\sum_{k=0}^{\infty} 2^{-|k-j|\delta} |g_k(x)|\right)^q\right)^{\frac{1}{q}} \le C_0 \left(\sum_{j=0}^{\infty} \sum_{k=0}^{\infty} 2^{-|k-j|q\delta} |g_k(x)|^q\right)^{\frac{1}{q}}$$

$$= C_0 \left(\sum_{k=0}^{\infty} |g_k(x)|^q \left(\sum_{j=0}^{\infty} 2^{-|k-j|q\delta}\right)\right)^{\frac{1}{q}}$$

$$\le C_2 \left(\sum_{k=0}^{\infty} |g_k(x)|^q\right)^{\frac{1}{q}}.$$

By taking L_p-norms of both terms it follows (46). The case $q = \infty$ is trivial.

If $1 \le p \le \infty$, then

$$\|G_j|L_p\| \le C_0 \sum_{k=0}^{\infty} 2^{-|k-j|\delta} \|g_k|L_p\|$$

and therefore (45) follows from the previous proven assertion if one just replaces $G_j(x)$ by $\|G_j|L_p\|$ and $g_j(x)$ by $\|g_j|L_p\|$. In case of $0 < p < 1$ we have

$$\int_{\mathbb{R}^n} |G_j(x)|^p \, dx \le C_0 \sum_{k=0}^{\infty} 2^{-|k-j|\delta p} \int_{\mathbb{R}^n} |g_k(x)|^p \, dx$$

and so we get by the previous proven, this time applied to $\int_{\mathbb{R}^n} |G_j(x)|^p \, dx$ instead of $G_j(x)$, $\int_{\mathbb{R}^n} |g_k(x)|^p \, dx$ instead of $g_j(x)$ and $\frac{q}{p}$ instead of q,

$$\left(\sum_{j=0}^{\infty} \left(\int_{\mathbb{R}^n} |G_j(x)|^p \, dx\right)^{\frac{q}{p}}\right)^{\frac{p}{q}} \le C_1 \left(\sum_{j=0}^{\infty} \left(\int_{\mathbb{R}^n} |g_j(x)|^p \, dx\right)^{\frac{q}{p}}\right)^{\frac{p}{q}},$$

which matches our desired result (45). □

Now we come back to the initial topic. Let $G_0(x) := (\Psi^* f)_a(x)$, $G_j(x) = 2^{js}(\psi_j^* f)_a(x)$ for $j \in \mathbb{N}$, $g_0(x) = (\Phi^* f)_a(x)$ and $g_k(x) = 2^{ks}(\varphi_k^* f)_a(x)$ for $k \in \mathbb{N}$. Then the conditions of lemma 2.4 follow from (43) and (44) and we obtain from (45) and (46), after slight modification, the desired inequalities (32) and (33).

Second Step: Let $\Psi, \psi \in \mathcal{S}(\mathbb{R}^n)$ with (31) be given. We want to show that there exists a constant $C > 0$ with

$$\|(\Psi^* f)_a|L_p\| + \left(\sum_{j=1}^{\infty} 2^{jsq}\|(\psi_j^* f)_a|L_p\|^q\right)^{\frac{1}{q}} \leq C\|(\Psi\hat{f})\check{}|L_p(E)\| + C\left(\sum_{j=1}^{\infty} 2^{jsq}\|(\psi(2^{-j}\cdot)\hat{f})\check{}|L_p(E)\|^q\right)^{\frac{1}{q}} \tag{47}$$

resp.

$$\|(\Psi^* f)_a|L_p\| + \left\|\left(\sum_{j=1}^{\infty} 2^{jsq}\left((\psi_j^* f)_a\right)^q\right)^{\frac{1}{q}}\Big|L_p\right\| \leq C\|(\Psi\hat{f})\check{}|L_p(E)\| + C\left\|\left(\sum_{j=1}^{\infty} 2^{jsq}\|(\psi(2^{-j}\cdot)\hat{f})\check{}|E\|^q\right)^{\frac{1}{q}}\Big|L_p\right\|. \tag{48}$$

At the beginning we choose once again $\Lambda, \lambda \in \mathcal{S}(\mathbb{R}^n)$ for our given $\Psi, \psi \in \mathcal{S}(\mathbb{R}^n)$ by lemma 2.2 with

$$\begin{aligned} supp\ \Lambda &\subset \{|x| < 2\varepsilon\}, \\ supp\ \lambda &\subset \left\{\frac{\varepsilon}{2} < |x| < 2\varepsilon\right\}, \\ 1 &= \Lambda(x)\Psi(x) + \sum_{k=1}^{\infty} \lambda(2^{-k}x)\psi(2^{-k}x). \end{aligned}$$

If we replace x by $2^{-j}x$ for $j \in \mathbb{N}$ in the last relation, it follows

$$1 = \Lambda(2^{-j}x)\Psi(2^{-j}x) + \sum_{k=j+1}^{\infty} \lambda(2^{-k}x)\psi(2^{-k}x)$$

and

$$f = \left(\check{\Lambda}_j * \check{\Psi}_j\right) * f + \sum_{k=j+1}^{\infty} \check{\lambda}_k * \left(\check{\psi}_k * f\right).$$

This yields

$$\left(\check{\psi}_j * f\right)(y) = \left(\left(\check{\Lambda}_j * \check{\Psi}_j\right) * \left(\check{\psi}_j * f\right)\right)(y) + \sum_{k=j+1}^{\infty} \left(\left(\check{\psi}_j * \check{\lambda}_k\right) * \left(\check{\psi}_k * f\right)\right)(y) \tag{49}$$

for all $y \in \mathbb{R}^n$. We deduce for all $N \in \mathbb{N}$ with lemma 2.3 ($k \geq j$)

$$\begin{aligned}\left|\left(\check{\psi}_j * \check{\lambda}_k\right)(z)\right| &= \left|2^{jn}\left(\check{\psi} * \check{\lambda}_{k-j}\right)(2^j z)\right| = \left|2^{jn}\left(\check{\psi} * \lambda(2^{j-k}\cdot)^{\check{}}\right)(2^j z)\right| \\ &\leq c_{\psi,\lambda,N}\,\frac{2^{jn}2^{(j-k)N}}{(1+2^j|z|)^a}\end{aligned}$$

(without using any moment conditions on ψ) and obviously

$$\begin{aligned}\left|\left(\check{\Psi}_j * \check{\Lambda}_j\right)(z)\right| &= 2^{jn}\left(\check{\Psi} * \check{\Lambda}\right)(2^j z) \\ &\leq c_{\Psi,\Lambda}\frac{2^{jn}}{(1+2^j|z|)^a},\end{aligned}$$

because if $f, g \in \mathcal{S}(\mathbb{R}^n)$, then $f * g \in \mathcal{S}(\mathbb{R}^n)$, too. If we insert these two estimates into (49), we obtain

$$\|\left(\check{\psi}_j * f\right)(y)|E\| \leq C_N \sum_{k=j}^{\infty} 2^{jn}2^{(j-k)N}\int_{\mathbb{R}^n}\frac{\|\left(\check{\psi}_k * f\right)(z)|E\|}{(1+2^j|y-z|)^a}\,dz \tag{50}$$

for all $f \in \mathcal{S}'(\mathbb{R}^n, E)$. Now we divide both sides by $(1+2^j|x-y|)^a$ and use the inequality

$$(1+2^j|x-z|)^a \leq (1+2^j|x-y|)^a(1+2^j|y-z|)^a$$

to get

$$(\psi_j^* f)_a(x) \leq C_N \sum_{k=j}^{\infty} 2^{jn}2^{(j-k)N}\int_{\mathbb{R}^n}\frac{\|\left(\check{\psi}_k * f\right)(z)|E\|}{(1+2^j|x-z|)^a}\,dz$$

because the right-hand side does not depend on y. Let $r \in (0,1]$ be fixed. Keeping in mind $k \geq j$ we arrive with

$$\begin{aligned}\frac{\|\left(\check{\psi}_k * f\right)(z)|E\|}{(1+2^j|x-z|)^a} &\leq \|\left(\check{\psi}_k * f\right)(z)|E\|^r\left[(\psi_k^* f)_a(x)\right]^{1-r}\frac{(1+2^k|x-z|)^{a(1-r)}}{(1+2^j|x-z|)^a} \\ &\leq \|\left(\check{\psi}_k * f\right)(z)|E\|^r\left[(\psi_k^* f)_a(x)\right]^{1-r}\frac{2^{a(k-j)}}{(1+2^k|x-z|)^{ar}}\end{aligned}$$

(see (26)) at

$$\begin{aligned}(\psi_j^* f)_a(x) &\leq C_N \sum_{k=j}^{\infty} 2^{jn}2^{(j-k)N}2^{a(k-j)}[(\psi_k^* f)_a(x)]^{1-r}\int_{\mathbb{R}^n}\frac{\|\left(\check{\psi}_k * f\right)(z)|E\|^r}{(1+2^k|x-z|)^{ar}}\,dz \\ &= C_N \sum_{k=j}^{\infty} 2^{(j-k)N'}[(\psi_k^* f)_a(x)]^{1-r}\int_{\mathbb{R}^n}\frac{2^{kn}\|\left(\check{\psi}_k * f\right)(z)|E\|^r}{(1+2^k|x-z|)^{ar}}\,dz,\end{aligned} \tag{51}$$

where $N' = N - a + n$ still can be chosen arbitrarily large. This relation holds in an analogous way for Ψ instead of ψ_j and we get slightly varied

$$\begin{aligned}(\Psi^* f)_a(x) &\leq C_N[(\Psi^* f)_a(x)]^{1-r}\int_{\mathbb{R}^n}\frac{\|\left(\check{\Psi} * f\right)(z)|E\|^r}{(1+|x-z|)^{ar}}\,dz \\ &\quad + C_N\sum_{k=1}^{\infty} 2^{-kN'}[(\psi_k^* f)_a(x)]^{1-r}\int_{\mathbb{R}^n}\frac{2^{kn}\|\left(\check{\psi}_k * f\right)(z)|E\|^r}{(1+2^k|x-z|)^{ar}}\,dz.\end{aligned}$$

We have to modify these two estimates a bit. For that reason we use a lemma which can be directly adopted from [Ryc99].

Lemma 2.5. *Let* $0 < r \leq 1$ *and* $\{b_j\}_{j=0}^{\infty}$, $\{d_j\}_{j=0}^{\infty}$ *be two sequences with values in* $(0, \infty]$ *resp.* $(0, \infty)$. *Let there be an* $N_0 \in \mathbb{N}$ *with*

$$\limsup_{j\to\infty} \frac{d_j}{2^{jN_0}} < \infty \tag{52}$$

and for all $N \in \mathbb{N}$ *a* $C_N > 0$ *such that*

$$d_j \leq C_N \sum_{k=j}^{\infty} 2^{(j-k)N} b_k d_k^{1-r} \qquad \textit{for } j \in \mathbb{N}_0$$

holds. Then for all $N \in \mathbb{N}$ *we have*

$$d_j^r \leq C_N \sum_{k=j}^{\infty} 2^{(j-k)Nr} b_k \qquad \textit{for } j \in \mathbb{N}_0$$

with the same constants C_N.

Proof. For $j \in \mathbb{N}_0$ set $D_{j,N} := \sup_{k\geq j} 2^{(j-k)N} d_k$. Then it follows

$$\begin{aligned} D_{j,N} &\leq C_N \sup_{k\geq j} 2^{(j-k)N} \sum_{l=k}^{\infty} 2^{(k-l)N} b_l d_l^{1-r} \\ &= C_N \sum_{l=j}^{\infty} 2^{(j-l)N} b_l d_l^{1-r} \\ &\leq C_N \sum_{l=j}^{\infty} 2^{(j-l)Nr} b_l D_{j,N}^{1-r}. \end{aligned}$$

If $D_{j,N} < \infty$, we have

$$d_j^r \leq D_{j,N}^r \leq C_N \sum_{l=j}^{\infty} 2^{(j-l)Nr} b_k.$$

By assumption (52) there exists an N_0 such that we have $D_{j,N} < \infty$ for all $N \geq N_0$. Hence our proposition is shown for these N's. Since the right-hand side decreases with N, the estimates follow for all $N < N_0$ as well, but with constant C_{N_0} instead of C_N. But now we have for $N < N_0$ and $k \geq j$

$$\begin{aligned} 2^{(j-k)N} d_k &\leq 2^{(j-k)N} C_{N_0}^{\frac{1}{r}} \left[\sum_{l=k}^{\infty} 2^{(k-l)Nr} b_l \right]^{\frac{1}{r}} \\ &= C_{N_0}^{\frac{1}{r}} \left[\sum_{l=k}^{\infty} 2^{(j-l)Nr} b_l \right]^{\frac{1}{r}} \\ &\leq C_{N_0}^{\frac{1}{r}} \left[\sum_{l=j}^{\infty} 2^{(j-l)Nr} b_l \right]^{\frac{1}{r}}. \end{aligned}$$

Either the right-hand side is infinity. Then our lemma is surely correct (set $j = k$). Or it is finite and thus $D_{j,N}$ as well. Hence we can apply the above argument for this case, too. □

For fixed $x \in \mathbb{R}^n$ we make use of lemma 2.5 with $d_j = (\psi_j^* f)_a(x)$ for $j \in \mathbb{N}$, $d_0 = (\Psi^* f)_a(x)$ and

$$b_j = \int_{\mathbb{R}^n} \frac{2^{kn} \| \left(\check{\psi}_k * f\right)(z) | E\|^r}{(1 + 2^k|x - z|)^{ar}} \, dz \text{ for all } j \in \mathbb{N}, \quad b_0 = \int_{\mathbb{R}^n} \frac{\| \left(\check{\Psi} * f\right)(z) | E\|^r}{(1 + |x - z|)^{ar}} \, dz.$$

We want to point out that we vary the procedure from [Ryc99] a bit here. We deal with the question whether the d_j fulfil condition (52) in the last step of the proof. The other conditions precisely result from the calculations above (see e.g. (51)). If applicable, we get

$$(\psi_j^* f)_a(x)^r \leq C_N' \sum_{k=j}^{\infty} 2^{(j-k)Nr} \int_{\mathbb{R}^n} \frac{2^{kn} \| \left(\check{\psi}_k * f\right)(z) | E\|^r}{(1 + 2^k|x - z|)^{ar}} \, dz \tag{53}$$

and

$$\begin{aligned}(\Psi^* f)_a(x)^r \leq C_N' &\int_{\mathbb{R}^n} \frac{\| \left(\check{\Psi} * f\right)(z) | E\|^r}{(1 + |x - z|)^{ar}} \, dz \\ &+ C_N' \sum_{k=1}^{\infty} 2^{-kNr} \int_{\mathbb{R}^n} \frac{2^{kn} \| \left(\check{\psi}_k * f\right)(z) | E\|^r}{(1 + 2^k|x - z|)^{ar}} \, dz\end{aligned} \tag{54}$$

with $C_N' = C_{N+a-n}$. Here C_N' does not depend on $f \in \mathcal{S}'(\mathbb{R}^n, E)$, $j \in \mathbb{N}$ or $r \in (0, 1]$.

We like to note that (53) in the case $r > 1$ can more easily be derived from (50) if we replace a by $a + n + 1$. By applying Hölder's inequality two times we arrive at

$$\begin{aligned}
&\| \left(\check{\psi}_j * f\right)(y) | E\| \\
&\leq C_N \sum_{k=j}^{\infty} 2^{jn} 2^{(j-k)N} \int_{\mathbb{R}^n} \frac{\| \left(\check{\psi}_k * f\right)(z) | E\|}{(1 + 2^j|y - z|)^{a+n+1}} \, dz \\
&\leq C_N \sum_{k=j}^{\infty} 2^{jn} 2^{(j-k)N} \|(1 + 2^j| \cdot - z|)^{-(n+1)} | L_{r'}\| \left(\int_{\mathbb{R}^n} \frac{\| \left(\check{\psi}_k * f\right)(z) | E\|^r}{(1 + 2^j|y - z|)^{ar}} \, dz \right)^{\frac{1}{r}} \\
&\leq \tilde{C}_N \sum_{k=j}^{\infty} 2^{j(n - \frac{n}{r'})} 2^{(j-k)N} \left(\int_{\mathbb{R}^n} \frac{\| \left(\check{\psi}_k * f\right)(z) | E\|^r}{(1 + 2^j|y - z|)^{ar}} \, dz \right)^{\frac{1}{r}} \\
&\leq \tilde{C}_N \left(\sum_{k=j}^{\infty} 2^{jn} 2^{(j-k)(N-1)r} \int_{\mathbb{R}^n} \frac{\| \left(\check{\psi}_k * f\right)(z) | E\|^r}{(1 + 2^j|y - z|)^{ar}} \, dz \right)^{\frac{1}{r}} \left(\sum_{k=j}^{\infty} 2^{(j-k)r'} \right)^{\frac{1}{r'}} \\
&\leq C_N' \left(\sum_{k=j}^{\infty} 2^{(j-k)(N-1+\frac{n}{r}-a)r} \int_{\mathbb{R}^n} \frac{2^{kn} \| \left(\check{\psi}_k * f\right)(z) | E\|^r}{(1 + 2^k|y - z|)^{ar}} \, dz \right)^{\frac{1}{r}}.
\end{aligned}$$

If we use the inequality

$$(1 + 2^j|x - z|)^a \leq (1 + 2^j|x - y|)^a (1 + 2^j|y - z|)^a$$

when dividing by $(1+2^j|x-y|)^a$, we get

$$(\psi_j^* f)_a(x) \le C'_N \left(\sum_{k=j}^{\infty} 2^{(j-k)(N-1+\frac{n}{r}-a)r} \int_{\mathbb{R}^n} \frac{2^{kn} \| \left(\check{\psi}_k * f\right)(z)|E\|^r}{(1+2^k|x-z|)^{ar}} \, dz \right)^{\frac{1}{r}}$$

and an analogous result for $(\Psi^* f)_a(x)$, which provides the desired results (53) and (54) - because $N \in \mathbb{N}$ was arbitrary - in case of $r > 1$.

By our assumptions on a we can choose r in such a way that $\frac{n}{a} < r < p$ resp. $\frac{n}{a} < r < \min(p,q)$. Then we have $h(x) := \frac{1}{(1+|x|)^{ar}} \in L_1$. Additionally $h(x) \ge 0$ for all $x \in \mathbb{R}^n$ and $h(x) < h(y)$ if $|x| > |y|$. If we denote by $M(g)$ the Hardy-Littlewood maximal function of the function $g \in L_1^{loc}$ introduced in (4), it follows from the majority property (see (5))

$$|\left(h_t * g\right)(x)| \le c\, M(g)(x)$$

for all $t > 0$. If we use this for (53) (and (54)) with $g(z) = \| \left(\check{\psi}_k * f\right)(z)|E\|^r$ and $N = \lfloor \max(-s,0) \rfloor + 1$, we come to

$$2^{jsr} (\psi_j^* f)_a(x)^r \le C'_N \sum_{k=j}^{\infty} 2^{(j-k)\delta} 2^{ksr} M \left(\| \left(\check{\psi}_k * f\right) |E\|^r \right)(x)$$

and

$$\begin{aligned}
&(\Psi^* f)_a(x)^r \\
&\quad \le C'_N M \left(\| \left(\check{\Psi} * f\right)(z)|E\|^r \right)(x) + C'_N \sum_{k=1}^{\infty} 2^{-k\delta} 2^{ksr} M \left(\| \left(\check{\psi}_k * f\right)(z)|E\|^r \right)(x)
\end{aligned}$$

for a suitable $\delta > 0$. Now we apply lemma 2.4 with $G_j = 2^{jsr}[(\psi_j^* f)_a]^r$ for $j \in \mathbb{N}$, $G_0 = [(\Psi^* f)_a]^r$, $g_k = 2^{ksr} M \left(\| \left(\check{\psi}_k * f\right)(z)\|^r \right)$ for $k \in \mathbb{N}$, $g_0 = M \left(\| \left(\check{\Psi} * f\right)(z)\|^r \right)$, $\tilde{q} = \frac{q}{r}$ and $\tilde{p} = \frac{p}{r}$. We obtain

$$\begin{aligned}
\|(\Psi^* f)_a|L_p\| + \left(\sum_{j=1}^{\infty} 2^{jsq} \|(\psi_j^* f)_a|L_p\|^q \right)^{\frac{1}{q}} &\le C \left\| M \left(\|(\Psi \hat{f})\check{}\,|E\|^r \right) |L_{\frac{p}{r}} \right\|^{\frac{1}{r}} \\
&+ C \left(\left(\sum_{j=1}^{\infty} 2^{jsq} \left\| M \left(\|(\psi(2^{-j}\cdot)\hat{f})\check{}\,|E\|^r \right) |L_{\frac{p}{r}} \right\|^{\frac{q}{r}} \right)^{\frac{r}{q}} \right)^{\frac{1}{r}}
\end{aligned}$$

resp.

$$\begin{aligned}
\|(\Psi^* f)_a|L_p\| + \left\| \left(\sum_{j=1}^{\infty} 2^{jsq} \left((\psi_j^* f)_a\right)^q \right)^{\frac{1}{q}} |L_p \right\| &\le C \left\| M \left(\|(\Psi \hat{f})\check{}\,|E\|^r \right) |L_{\frac{p}{r}} \right\|^{\frac{1}{r}} \\
&+ C \left\| \left(\sum_{j=1}^{\infty} \left[M \left(2^{jsr} \|(\psi(2^{-j}\cdot)\hat{f})\check{}\,|E\|^r \right) \right]^{\frac{q}{r}} \right)^{\frac{r}{q}} |L_{\frac{p}{r}} \right\|^{\frac{1}{r}}.
\end{aligned}$$

Because $\frac{p}{r} > 1$ and in case of $F^s_{p,q}(E)$ as well $\frac{q}{r} > 1$ (and $\frac{p}{r} < \infty$) it follows from the boundedness of the maximal operator from $L_{\frac{p}{r}}$ to $L_{\frac{p}{r}}$ resp. from $L_{\frac{p}{r}}(l_{\frac{q}{r}})$ to $L_{\frac{p}{r}}(l_{\frac{q}{r}})$ (see (6) resp. (7))

$$\|(\Psi^* f)_a|L_p\| + \left(\sum_{j=1}^{\infty} 2^{jsq}\|(\psi_j^* f)_a|L_p\|^q\right)^{\frac{1}{q}} \le C' \left\| \, \|(\Psi\hat{f})\check{}|E\|^r \; |L_{\frac{p}{r}}\right\|^{\frac{1}{r}}$$
$$+ C' \left(\left(\sum_{j=1}^{\infty} 2^{jsq} \left\| \, \|(\psi(2^{-j}\cdot)\hat{f})\check{}|E\|^r \; |L_{\frac{p}{r}} \right\|^{\frac{q}{r}} \right)^{\frac{r}{q}} \right)^{\frac{1}{r}}$$

resp.

$$\|(\Psi^* f)_a|L_p\| + \left\| \left(\sum_{j=1}^{\infty} 2^{jsq}\left((\psi_j^* f)_a\right)^q\right)^{\frac{1}{q}} |L_p\right\| \le C' \left\| \, \|(\Psi\hat{f})\check{}|E\|^r \; |L_{\frac{p}{r}}\right\|^{\frac{1}{r}}$$
$$+ C' \left\| \left(\sum_{j=1}^{\infty} \left[2^{jsr} \|(\psi(2^{-j}\cdot)\hat{f})\check{}|E\|^r \right]^{\frac{q}{r}} \right)^{\frac{r}{q}} |L_{\frac{p}{r}} \right\|^{\frac{1}{r}},$$

which matches (at close view) our desired results (47) and (48).

Third step: Now we will conclude the equivalences of the norms $\|\cdot|B^s_{p,q}(E)\|$, $\|\cdot|B^s_{p,q}(E)\|_{\Psi,\psi}$ and $\|\cdot|B^s_{p,q}(E)\|^*_{\Psi,\psi}$ by the results of the first and the second step. We choose a smooth dyadic resolution of unity consisting of the functions $\Phi = \varphi_0$ and $\{\varphi_j\}_{j\in\mathbb{N}}$ with $\varphi_j(\cdot) = \varphi(2^{-j}\cdot)$ (see (14)) such that

$$\varphi_0(\xi) > 0 \text{ for } |\xi| < 2,$$
$$\varphi(\xi) > 0 \text{ for } \frac{1}{2} < |\xi| < 2.$$

Obviously it holds

$$\|(\Psi\hat{f})\check{}|L_p(E)\| + \left(\sum_{j=1}^{\infty} 2^{jsq}\|(\psi(2^{-j}\cdot)\hat{f})\check{}|L_p(E)\|^q\right)^{\frac{1}{p}}$$
$$\le \|(\Psi^* f)_a|L_p\| + \left(\sum_{j=1}^{\infty} 2^{jsq}\|(\psi_j^* f)_a|L_p\|^q\right)^{\frac{1}{p}}.$$

By the first step (see (32)) we get

$$\|(\Psi^* f)_a|L_p\| + \left(\sum_{j=1}^{\infty} 2^{jsq}\|(\psi_j^* f)_a|L_p\|^q\right)^{\frac{1}{p}}$$
$$\le C\|(\Phi^* f)_a|L_p\| + C\left(\sum_{j=1}^{\infty} 2^{jsq}\|(\varphi_j^* f)_a|L_p\|^q\right)^{\frac{1}{p}}$$

because Φ and φ fulfil the necessary conditions for $\varepsilon' = 1$. Now it follows by the second step, applied to Φ and φ,

$$\begin{aligned}
\|(\Phi^* f)_a|L_p\| + &\left(\sum_{j=1}^{\infty} 2^{jsq}\|(\varphi_j^* f)_a|L_p\|^q\right)^{\frac{1}{p}} \\
&\leq C'\|(\Phi\hat{f})\check{}|L_p(E)\| + C'\left(\sum_{j=1}^{\infty} 2^{jsq}\|(\varphi(2^{-j}\cdot)\hat{f})\check{}|L_p(E)\|^q\right)^{\frac{1}{q}} \\
&= C'\|f|B^s_{p,q}(E)\|
\end{aligned}$$

if our not yet proven condition of finiteness on d_j in lemma 2.5 is true. We will turn our attention to this question immediately.

Otherwise we obtain from the first step - this time by interchanging the roles of φ and ψ resp. Φ and Ψ (this can be done, because $D^\alpha\varphi(0) = 0$ for all $\alpha \in \mathbb{N}^n$) - and from the second step, applied to Ψ and ψ,

$$\begin{aligned}
\|f|B^s_{p,q}(E)\| &\leq \|(\Phi^* f)_a|L_p\| + \left(\sum_{j=1}^{\infty} 2^{jsq}\|(\varphi_j^* f)_a|L_p\|^q\right)^{\frac{1}{p}} \\
&\leq C\|(\Psi^* f)_a|L_p\| + C\left(\sum_{j=1}^{\infty} 2^{jsq}\|(\psi_j^* f)_a|L_p\|^q\right)^{\frac{1}{p}} \\
&\leq C'\|(\Psi\hat{f})\check{}|L_p(E)\| + C'\left(\sum_{j=1}^{\infty} 2^{jsq}\|(\psi(2^{-j}\cdot)\hat{f})\check{}|L_p(E)\|^q\right)^{\frac{1}{q}}
\end{aligned}$$

if our not yet proven condition of finiteness on d_j in lemma 2.5 is true.

Now let's take a look at this condition: Let us at first assume f to be in $B^s_{p,q}(E)$. Then by the lift property (see (17)) and the Sobolev embeddings (see (18)) there is a $\sigma \in \mathbb{R}^+$ such that $g := ((1+|\xi|^2)^{-\sigma}\hat{f})\check{} \in L_\infty(E)$. We obtain

$$\begin{aligned}
d_j = (\psi_j^* f)_a(x) &= \sup_{y\in\mathbb{R}^n} \frac{\left\|\left(\psi(2^{-j}\cdot)\hat{f}\right)\check{}(y)\middle|E\right\|}{(1+2^j|x-y|)^a} \\
&= \sup_{y\in\mathbb{R}^n} \frac{\left\|\left(\psi(2^{-j}\cdot)(1+|\xi|^2)^{+\sigma}(1+|\xi|^2)^{-\sigma}\hat{f}\right)^{\vee}(y)\middle|E\right\|}{(1+2^j|x-y|)^a} \\
&= \sup_{y\in\mathbb{R}^n} \frac{\left\|\left((\psi(2^{-j}\cdot)(1+|\xi|^2)^\sigma)\check{} * g\right)(y)\middle|E\right\|}{(1+2^j|x-y|)^a} \\
&\leq \|g|L_\infty(E)\| \cdot \|(\psi(2^{-j}\cdot)(1+|\xi|^2)^\sigma)\check{}|L_1\| \\
&\leq C\|g|L_\infty(E)\| \cdot \|\check{\psi}_j|B^{2\sigma}_{1,1}\| \\
&\leq C'\|g|L_\infty(E)\|2^{jn}2^{j(2\sigma-n)}\|\check{\psi}|B^{2\sigma}_{1,1}\|,
\end{aligned}$$

where the last estimate is due to [Tri92], section 2.3.3, p. 100.[3] Therefore, we get the requested condition (52) with $N_0 = \lceil 2\sigma \rceil$ and the desired result follows. The proof in the $F^s_{p,q}(E)$-case is the same.

[3]One could have derived this in a more elementary way by using $D^\alpha\hat{f} = (-i)^{|\alpha|}\widehat{x^\alpha f}$.

Fourth step: Last but not least we show the characterizations (27) and (28) for $B^s_{p,q}(E)$. The proof of the $F^s_{p,q}(E)$-case is the same.

In the first step we didn't use the condition $f \in B^s_{p,q}(E)$. So, if for an $f \in S'(\mathbb{R}^n, E)$ we have $\|f|B^s_{p,q}(E)\| < \infty$, then $\|f|B^s_{p,q}(E)\|^a_{\psi,\Psi} < \infty$ for admissible a and vice versa. Therefore we have (28).

In the second step we used the condition $f \in B^s_{p,q}(E)$ only in lemma 2.5 for fulfilling (52). If we just assume $f \in \mathcal{S}'(\mathbb{R}^n, E)$ instead, then there exist constants $c > 0, K, L \in \mathbb{N}_0$ such that for all $\varphi \in \mathcal{S}(\mathbb{R}^n)$ it holds

$$\| \left(\check{\varphi} * f\right)(x)|E\| \leq c(1+|x|^2)^{\frac{K}{2}} \|\check{\varphi}\|_{K,L},$$

where $\|\cdot\|_{K,L}$ is the seminorm introduced in (8). Hence it follows if $a \geq K$

$$\begin{aligned} d_j = (\psi^*_j f)_a(x) &= \sup_{y\in\mathbb{R}^n} \frac{\| \left(\check{\psi}_j * f\right)(x-y)|E\|}{(1+2^j|y|)^a} \\ &\leq \sup_{y\in\mathbb{R}^n} \frac{(1+|x-y|^2)^{\frac{K}{2}}}{(1+2^j|y|)^a} \|\check{\psi}_j\|_{K,L} \\ &\leq c'(1+|x|^2)^{\frac{K}{2}} 2^{j(L+n)} \|\check{\psi}\|_{K,L}, \end{aligned}$$

where c' is independent of j and x. So the conditions of lemma 2.5 are fulfilled for "large" a with $N_0 = L + n$. Thus it follows that if $\|f|B^s_{p,q}(E)\|_{\psi,\Psi}$ is finite, then as well $\|f|B^s_{p,q}(E)\|^a_{\psi,\Psi}$ is finite for these a and hence $f \in B^s_{p,q}(E)$ by the third step. If $\|f|B^s_{p,q}(E)\| < \infty$, then $\|f|B^s_{p,q}(E)\|^a_{\psi,\Psi}$ is finite for admissible a by the first step and the end of the third step and therefore obviously $\|f|B^s_{p,q}(E)\|_{\psi,\Psi}$, too. □

Remark 2.6. If we take a closer look at the proof, we can see that we didn't use the conditions (23) and (24) in the first step. Hence we can apply the first chain of inequalities even without them. Thus it holds

$$\|(\Psi^* f)_a|L_p\| + \left(\sum_{j=1}^{\infty} 2^{jsq} \|(\psi^*_j f)_a|L_p\|^q\right)^{\frac{1}{q}} \leq C\|f|B^s_{p,q}(E)\|,$$

$$\|(\Psi^* f)_a|L_p\| + \left\| \left(\sum_{j=1}^{\infty} 2^{jsq} \left((\psi^*_j f)_a\right)^q\right)^{\frac{1}{q}} |L_p \right\| \leq C\|f|F^s_{p,q}(E)\|$$

for a suitable constant $C > 0$.

In the second step we didn't use the moment condition (25). So we can apply the second chain of inequalities even without this assumption, because the application of the first step depends only on the moment conditions of φ. Hence it follows even without this assumption

$$\|f|B^s_{p,q}(E)\| \leq C\|(\Psi\hat{f})\check{}|L_p(E)\| + \left(\sum_{j=1}^{\infty} 2^{jsq} \|(\psi(2^{-j}\cdot)\hat{f})\check{}|L_p(E)\|^q\right)^{\frac{1}{q}},$$

$$\|f|F^s_{p,q}(E)\| \leq \|(\Psi\hat{f})\check{}|L_p(E)\| + \left\| \left(\sum_{j=1}^{\infty} 2^{jsq} \|(\psi(2^{-j}\cdot)\hat{f})\check{}|E\|^q\right)^{\frac{1}{q}} |L_p \right\|.$$

From the above theorem and its proof follows a theorem on **continuous** versions of the norm by slight, but technically complex modifications. An example for such norms is given in [Tri92], section 2.4.1, p. 101 and section 2.4.3, p. 115.

Proposition 2.7. *Let* $d > 0$.

(i) Under the assumptions of theorem 2.1, *part (i) for* Ψ *and* ψ

$$\|f|B^s_{p,q}(E)\|^I_{\Psi,\psi} := \|(\Psi\hat{f})\check{}|L_p(E)\| + \left(\int_0^1 t^{-sq}\|(\psi(t\cdot)\hat{f})\check{}|L_p(E)\|^q \frac{dt}{t}\right)^{\frac{1}{q}}$$

and

$$\|f|B^s_{p,q}(E)\|^{\sup}_{\Psi,\psi} := \|\sup_{|\cdot-y|\le d}\|(\Psi\hat{f})\check{}(\cdot)|E\|\ |L_p\| + \left(\int_0^1 t^{-sq}\|\sup\|(\psi(\tau\cdot)\hat{f})\check{}|E\|\ |L_p\|^q \frac{dt}{t}\right)^{\frac{1}{q}}$$

are equivalent norms for $\|\cdot|B^s_{p,q}(E)\|$, *where* sup *is the supremum taken over* $\{|x-y|\le dt, t\le\tau\le 2t\}$ *for a fixed* $x\in\mathbb{R}^n$. *It holds*

$$B^s_{p,q}(E) = \left\{f\in\mathcal{S}'(\mathbb{R}^n,E) : \|f|B^s_{p,q}(E)\|^I_{\Psi,\psi} < \infty\right\}$$

and

$$B^s_{p,q}(E) = \left\{f\in\mathcal{S}'(\mathbb{R}^n,E) : \|f|B^s_{p,q}(E)\|^{\sup}_{\Psi,\psi} < \infty\right\}.$$

(ii) Under the assumptions of theorem 2.1, *part (ii) for* Ψ *and* ψ

$$\|f|F^s_{p,q}(E)\|^I_{\Psi,\psi}\|(\Psi\hat{f})\check{}|L_p(E)\| + \left\|\left(\int_0^1 t^{-sq}\|(\psi(t\cdot)\hat{f})\check{}|E\|^q \frac{dt}{t}\right)^{\frac{1}{q}} |L_p\right\| \tag{55}$$

and

$$\|f|F^s_{p,q}(E)\|^{\sup}_{\Psi,\psi} := \|\sup_{|\cdot-y|\le d}\|(\Psi\hat{f})\check{}(\cdot)|E\|\ |L_p\| + \left\|\left(\int_0^1 t^{-sq}\sup\|(\psi(\tau\cdot)\hat{f})\check{}|E\|^q\right)^{\frac{1}{q}} |L_p\right\| \tag{56}$$

are equivalent norms for $\|\cdot|B^s_{p,q}(E)\|$, *where* sup *is the supremum taken over* $\{|x-y|\le dt, t\le\tau\le 2t\}$ *for a fixed* $x\in\mathbb{R}^n$. *It holds*

$$F^s_{p,q}(E) = \left\{f\in\mathcal{S}'(\mathbb{R}^n,E) : \|f|F^s_{p,q}(E)\|^I_{\Psi,\psi} < \infty\right\} \tag{57}$$

and

$$F^s_{p,q}(E) = \left\{f\in\mathcal{S}'(\mathbb{R}^n,E) : \|f|F^s_{p,q}(E)\|^{\sup}_{\Psi,\psi} < \infty\right\}. \tag{58}$$

Proof. We restrict ourselves to the case of the $F^s_{p,q}(E)$-spaces, the $B^s_{p,q}(E)$-case can be treated in an analogous way. For this purpose we first consider (56), which is obviously larger than (55), and show that we can estimate this term by $C\|f|F^s_{p,q}(E)\|$. On that account we look back at the first step of the proof of theorem 2.1. But this time we start with $\psi(\tau\cdot)$ with $1 \le \tau \le 4$ instead of ψ. For given Φ and φwe again choose associated Λ and λ by lemma 2.2. We argue in the same way as in (36) and (37) and obtain

$$\| \left(\psi(\tau\cdot)\check{}_j * \check{\lambda}_k * \check{\varphi}_k * f\right)(y)|E\| \le (\varphi^*_k f)_a(y) \int_{\mathbb{R}^n} \left|\left(\psi(\tau\cdot)\check{}_j * \check{\lambda}_k\right)(z)\right| (1+2^k|z|)^a \, dz.$$

Thereby observe

$$\psi(\tau\cdot)\check{}_j = \psi(\tau 2^{-j}\cdot)\check{}.$$

Now we apply lemma 2.3 as in step 1 (using $1 \le \tau \le 4$). For $k \le j$ we get with the substitution $2^k z \to z$

$$\begin{aligned}
\int\limits_{\mathbb{R}^n} \left|\left(\psi(\tau\cdot)\check{}_j * \check{\lambda}_k\right)(z)\right| (1+2^k|z|)^a \, dz &= \int\limits_{\mathbb{R}^n} 2^{kn} \left|\left(\psi(\tau\cdot)\check{}_{j-k} * \check{\lambda}\right)(2^k z)\right| (1+2^k|z|)^a \, dz \\
&= \int_{\mathbb{R}^n} \left|\left(\psi(\tau\cdot)\check{}_{j-k} * \check{\lambda}\right)(z)\right| \cdot (1+|z|)^a \, dz \\
&\le c \sup_{z\in\mathbb{R}^n} \left|\left(\psi(2^{k-j}\tau\cdot)\check{} * \check{\lambda}\right)(z)\right| \cdot (1+|z|)^{a+n+1} \\
&\le c' \tau^{S+1} 2^{(k-j)(S+1)} \\
&\le c'' 2^{(k-j)(S+1)}.
\end{aligned}$$

In case of $k \ge j$ we obtain by the substitution $2^j\tau^{-1}y \to y$

$$\begin{aligned}
\left(\psi(\tau\cdot)\check{}_j * \check{\lambda}_k\right)(z) &= \int_{\mathbb{R}^n} \tau^{-n} 2^{(k+j)n} \check{\psi}\left(2^j\tau^{-1}(z-y)\right) \check{\lambda}(2^k y) \, dy \\
&= \int_{\mathbb{R}^n} 2^{kn} \check{\psi}(2^j\tau^{-1}z - y)\check{\lambda}(2^{k-j}\tau y) \, dy \\
&= \tau^{-n} 2^{jn} \left(\check{\psi} * \lambda(2^{j-k}\tau^{-1}\cdot)\check{}\right)(2^j\tau^{-1}z)
\end{aligned}$$

and consequently it follows by an analogous calculation as in the proof of the theorem

$$\begin{aligned}
&\int_{\mathbb{R}^n} \left|\left(\psi(\tau\cdot)\check{}_j * \check{\lambda}_k\right)(z)\right| \cdot (1+2^k|z|)^a \, dz \\
&\qquad = \int_{\mathbb{R}^n} \tau^{-n} 2^{jn} \left|\left(\check{\psi} * \lambda(2^{j-k}\tau^{-1}\cdot)\check{}\right)(2^j\tau^{-1}z)\right| \cdot (1+2^k|z|)^a \, dz \\
&\qquad = \int_{\mathbb{R}^n} \left|\left(\check{\psi} * \lambda(2^{j-k}\tau^{-1}\cdot)\check{}\right)(z)\right| \cdot (1+2^{k-j}\tau|z|)^a \, dz \\
&\qquad \le \tau^a 2^{(k-j)a} \int_{\mathbb{R}^n} \left|\left(\check{\psi} * \lambda(2^{j-k}\tau^{-1}\cdot)\check{}\right)(z)\right| \cdot (1+|z|)^a \, dz \\
&\qquad \le c_M \tau^a 2^{(k-j)a} \tau^{-(M+1)} 2^{(j-k)(M+1)} \\
&\qquad \le c'_M 2^{(k-j)a} 2^{(j-k)(M+1)},
\end{aligned}$$

where c'_M and c'' do not depend on τ. Hence this results in a counterpart of (41)

$$\sup_{y\in\mathbb{R}^n} \frac{\| \left(\psi(\tau\cdot)\check{}_j * \check{\lambda}_k * \check{\varphi}_k * f\right)(y)|E\|}{(1+2^j|x-y|)^a} \le C_{\psi,\lambda}\ \varphi_k^*(x) \begin{cases} 2^{(k-j)(S+1)} & ,k\le j \\ 2^{(j-k)(-s+1)} & ,k\ge j \end{cases}$$

independent of $\tau\in[1,4]$. We again come to

$$2^{js}\sup_{y\in\mathbb{R}^n} \frac{\| \left(\psi(\tau\cdot)\check{}_j * f\right)(y)|E\|}{(1+2^j|x-y|)^a} \le C2^{-j\delta}(\Phi^* f)_a(x) + C\sum_{k=1}^\infty 2^{ks}(\varphi_k^* f)_a(x)2^{-|j-k|\delta}$$

with $\delta=\min(S+1-s,1)>0$ and (taken over from step 1)

$$\sup_{y\in\mathbb{R}^n} \frac{\| \left(\check{\Psi}_j * f\right)(y)|E\|}{(1+|x-y|)^a} \le C(\Phi^* f)_a(x) + C\sum_{k=1}^\infty 2^{ks}(\varphi_k^* f)_a(x)2^{-k\delta}.$$

If we restrict each supremum to the domain $|x-y|\le d2^{-j+1}$ and use that for these y the inequality $(1+2^j|x-y|)^a\le c_d$ with a constant $c_d>0$ independent of j holds, we get

$$2^{js}\sup_{\substack{|x-y|\le d2^{-j+1},\\ 1\le\tau\le 4}} \| \left(\psi(\tau\cdot)\check{}_j * f\right)(y)|E\| \le C'2^{-j\delta}(\Phi^* f)_a(x) + C'\sum_{k=1}^\infty 2^{ks}(\varphi_k^* f)_a(x)2^{-|j-k|\delta}$$

and

$$\sup_{|x-y|\le d} \| \left(\check{\Psi}_j * f\right)(y)|E\| \le C'(\Phi^* f)_a(x) + C'\sum_{k=1}^\infty 2^{ks}(\varphi_k^* f)_a(x)2^{-k\delta}.$$

By applying lemma 2.4 as in step 1 this yields

$$\left\| \sup_{|\cdot-y|\le d} \|(\Psi\hat{f})\check{}(\cdot)|E\| \,\big|L_p\right\| + \left\| \left(\sum_{j=1}^\infty 2^{jsq} \sup_{\substack{|x-y|\le d2^{-j+1},\\ 1\le\tau\le 4}} \|(\psi(2^{-j}\tau\cdot)\hat{f})\check{}|E\|^q \right)^{\frac{1}{q}} \big|L_p\right\|$$
$$\le c\|(\Phi^* f)_a|L_p\| + c\left\| \left(\sum_{j=1}^\infty 2^{jsq}(\varphi_j^* f)_a)^q\right)^{\frac{1}{q}} \big|L_p\right\|.$$

But now we have for all $j\in\mathbb{N}$

$$\int_{2^{-j}}^{2^{-j+1}} t^{-sq}\sup \|(\psi(t\cdot)\hat{f})\check{}|E\|^q\ \frac{dt}{t} \le c_0 2^{jsq} \sup_{\substack{|x-y|\le d2^{-j+1},\\ 1\le\tau\le 4}} \|(\psi(2^{-j}\tau\cdot)\hat{f})\check{}|E\|^q,$$

where sup is the supremum for a fixed $x\in\mathbb{R}^n$ over $\{|x-y|\le dt, t\le\tau\le 2t\}$. If we take the sum over j of the integrals, we obtain

$$\left\| \sup_{|\cdot-y|\le d} \|(\Psi\hat{f})\check{}(\cdot)|E\| \,\big|L_p\right\| + \left\| \left(\int_0^1 t^{-sq}\sup \|(\psi(t\cdot)\hat{f})\check{}|E\|^q \right)^{\frac{1}{q}} \big|L_p\right\|$$
$$\le c'\|(\Phi^* f)_a|L_p\| + c'\left\| \left(\sum_{j=1}^\infty 2^{jsq}(\varphi_j^* f)_a)^q\right)^{\frac{1}{q}} \big|L_p\right\|$$

and so the norms (55) and (56) are estimated from above by $c'\|f|F^s_{p,q}(E)\|$.[4]

In the second part of the proof we want to estimate $\|f|F^s_{p,q}(E)\|$ from above again. For this we go back to step 1 of the proof of theorem 2.1, but this time interchanging the roles of Φ and Ψ and of φ and $\psi(\tau\cdot)$ in comparison to the just shown (see step 3 of the proof of theorem 2.1 for details). For given $\tau\in[1,2]$, Ψ and $\psi(\tau\cdot)$ we choose functions Λ^τ and λ^τ by lemma 2.2 with the properties (34) and (35), where ε' and therefore the supports of the functions λ^τ depend on τ in an obvious way. By looking at the construction in lemma 2.2 one can see that we can choose λ^τ to be $\lambda(\tau\cdot)$ and that for all $N,M\in\mathbb{N}$ there exists a $C_{N,M}$ such that

$$\sup_{y\in\mathbb{R}^n}(1+|y|)^N\sum_{|\alpha|\le M}|D^\alpha\Lambda^\tau(y)|\le C_{N,M},$$

i.e. that the $\mathcal{S}(\mathbb{R}^n)$-seminorms from (8) can be estimated uniformly in τ. This holds for λ^τ as well by the above remark. So we obtain the analogue of (36), with exchanged roles,

$$\left(\check{\varphi}_j*f\right)(y)=\left(\left(\check{\varphi}_j*\check{\Lambda^\tau}\right)*\left(\check{\Psi}*f\right)\right)(y)+\sum_{k=1}^{\infty}\left(\left(\check{\varphi}_j*\check{\lambda^\tau}_k\right)*\left(\psi(\tau\cdot)\check{}_k*f\right)\right)(y)$$

for all $y\in\mathbb{R}^n$. If we now follow the proof of step 1, we have to estimate the integrals from (37) as in (40) by a constant independent of τ. These are of the form

$$I^\tau_{j,k}:=\int_{\mathbb{R}^n}\left|\left(\check{\varphi}_j*\check{\lambda^\tau}_k\right)(z)\right|(1+|2^kz|)^a\,dz$$

and

$$\int_{\mathbb{R}^n}\left|\left(\check{\varphi}_j*\check{\Lambda^\tau}\right)(z)\right|(1+|z|)^a\,dz.$$

To estimate the integrals from above we used lemma 2.3. Please note that the constants appearing in this lemma only depend on the $\mathcal{S}(\mathbb{R}^n)$-seminorms and the behaviour at 0 of φ and λ resp. only of the $\mathcal{S}(\mathbb{R}^n)$-seminorms of Λ.[5] Hence there exists a constant independent of τ such that

$$I^\tau_{j,k}\le C_{\lambda,\varphi}\begin{cases}2^{(k-j)(S+1)} & ,k\le j\\ 2^{(j-k)(a-s+1)} & ,k\ge j\end{cases}.$$

There is an analogous result for Λ. If we go on with step 1 of the proof of theorem 2.1, we get the corresponding results of (41) and (42), with exchanged roles. Hence it holds

$$\sup_{y\in\mathbb{R}^n}\frac{\|\left(\check{\varphi}_j*\check{\lambda^\tau}_k*\psi(\tau\cdot)\check{}_k*f\right)(y)|E\|}{(1+2^j|x-y|)^a}\le C_{\lambda,\varphi}(\psi(\tau\cdot)^*_kf)_a(x)\begin{cases}2^{(k-j)(S+1)} & ,k\le j\\ 2^{(j-k)(-s+1)} & ,k\ge j\end{cases}$$

[4] It would have been enough to do the above calculations for one special $a>\frac{n}{p}$ resp. $\frac{n}{\min(p,q)}$ because the result does not depend on a.

[5] In the first part of this proof we could have used the same argumentation. But we decided to calculate it directly because of the easy form.

and

$$\sup_{y\in\mathbb{R}^n} \frac{\| \left(\check{\varphi}_j * \check{\Lambda}^\tau * \check{\Psi} * f\right)(y)|E\|}{(1+2^j|x-y|)^a} \le C_{\Lambda,\Psi}(\Psi^* f)_a(x) 2^{-j(S+1)}$$

with C independent of $\tau \in [1,2]$. Note that

$$\begin{aligned}(\psi(\tau\cdot)_k^* f)_a(x) &= \sup_{y\in\mathbb{R}^n} \frac{\|(\psi(2^{-k}\tau\cdot)\hat{f})\check{}(x-y)|E\|}{(1+2^k|y|)^a} \\ &\le c \sup_{y\in\mathbb{R}^n} \frac{\|(\psi(2^{-k}\tau\cdot)\hat{f})\check{}(x-y)|E\|}{(1+2^k\tau^{-1}|y|)^a} =: (\psi^{*'}_{2^{-k}\tau} f)_a(x).\end{aligned}$$

With the same steps as in the proof of theorem 2.1 we obtain as a result

$$2^{js}(\varphi_j^* f)_a(x) \le C 2^{-j\delta}(\Psi^* f)_a(x) + C\sum_{k=1}^\infty 2^{ks}(\psi^{*'}_{2^{-k}\tau} f)_a(x) 2^{-|j-k|\delta}$$

and

$$(\Phi^* f)_a(x) \le C(\Psi^* f)_a(x) + C\sum_{k=1}^\infty 2^{ks}(\psi^{*'}_{2^{-k}\tau} f)_a(x) 2^{-k\delta}$$

for a certain $\delta > 0$ and with C independent of τ. We raise the right-hand side to the power of q, integrate over $1 \le \tau \le 2$ with respect to $d\tau$, replace τ by t and take the q-th root. This yields

$$2^{js}(\varphi_j^* f)_a(x) \le C 2^{-j\delta}(\Psi^* f)_a(x) + C\left(\int_1^2 \left(\sum_{k=1}^\infty 2^{ks}2^{-|j-k|\delta}(\psi^{*'}_{2^{-k}t} f)_a(x)\right)^q dt\right)^{\frac{1}{q}}$$

and an analogous result for $(\Phi^* f)_a(x)$. In case of $q < 1$ it holds for $\{a_k(\cdot)\}_{k\in\mathbb{N}}$ with $a_k(\cdot) \ge 0$

$$\begin{aligned}\left\|\sum_{k=1}^\infty a_k|L_q\right\| &\le \left\|\left(\sum_{k=1}^\infty a_k^q\right)^{\frac{1}{q}}|L_q\right\| = \left(\sum_{k=1}^\infty \|a_k^q|L_1\|\right)^{\frac{1}{q}} \\ &\le c\sup_{k\in\mathbb{N}} 2^{|j-k|\varepsilon}\|a_k^q|L_1\|^{\frac{1}{q}} \le c\sum_{k=1}^\infty 2^{|j-k|\varepsilon}\|a_k|L_q\|\end{aligned}$$

for every $\varepsilon > 0$ and a suitable $c > 0$ depending on ε. We apply this to $a_k = 2^{ks}2^{-|j-k|\delta}(\psi^{*'}_{2^{-k}t} f)_a(x)$ and obtain

$$2^{js}(\varphi_j^* f)_a(x) \le C' 2^{-j\delta_0}(\Psi^* f)_a(x) + C'\sum_{k=1}^\infty 2^{-|j-k|\delta_0}2^{ks}\left(\int_1^2 \left((\psi^{*'}_{2^{-k}t} f)_a(x)\right)^q dt\right)^{\frac{1}{q}}$$

for a suitable $\delta_0 > 0$. In the case $q \ge 1$ this relation follows from the triangle inequality for L_q with $\delta_0 = \delta$. There is an analogous result for $(\Phi^* f)_a(x)$ again. Now we use lemma 2.4 as in the proof of theorem 2.1, but this time with

$$g_k := 2^{ks}\left(\int_1^2 \left((\psi^{*'}_{2^{-k}t} f)_a(x)\right)^q dt\right)^{\frac{1}{q}}$$

and conclude

$$\begin{aligned}
&\|(\Phi^* f)_a|L_p\| + \left\| \left(\sum_{j=1}^{\infty} 2^{jsq} \left((\varphi_j^* f)_a\right)^q \right)^{\frac{1}{q}} \Big| L_p \right\| \\
&\quad \le C\|(\Psi^* f)_a|L_p\| + C \left\| \left(\sum_{j=1}^{\infty} 2^{jsq} \int_1^2 \left((\psi^{*'}_{2^{-j}t} f)_a\right)^q dt \right)^{\frac{1}{q}} \Big| L_p \right\| \\
&\quad \le C'\|(\Psi^* f)_a|L_p\| + C' \left\| \left(\sum_{j=1}^{\infty} 2^{jsq} \int_{2^{-j}}^{2^{-j+1}} \left((\psi^{*'}_t f)_a\right)^q \frac{dt}{t} \right)^{\frac{1}{q}} \Big| L_p \right\| \\
&\quad \le C''\|(\Psi^* f)_a|L_p\| + C'' \left\| \left(\int_0^1 t^{-sq} \left((\psi^{*'}_t f)_a\right)^q \frac{dt}{t} \right)^{\frac{1}{q}} \Big| L_p \right\| .
\end{aligned} \tag{59}$$

Now we modify step 2 by applying it to $\psi(\tau\cdot)$ instead of ψ for $1 \le \tau \le 2$ to replace $(\psi^{*'}_t f)_a$ by $\psi(t\cdot)$. Here we can choose a to be arbitrarily large because it suffices to estimate one of the norms from above by the already shown equivalence of the norms from (29) and (30).

After choosing $\lambda, \Lambda \in \mathcal{S}(\mathbb{R}^n)$ with the desired properties

$$\begin{aligned}
supp\ \Lambda &\subset \{|x| < 2\varepsilon\}, \\
supp\ \lambda &\subset \left\{\frac{\varepsilon}{2} < |x| < 2\varepsilon\right\}, \\
1 = \Lambda(x)\Psi(x) &+ \sum_{k=1}^{\infty} \lambda(2^{-k}x)\psi(2^{-k}x)
\end{aligned}$$

it follows

$$1 = \Lambda(\tau x)\Psi(\tau x) + \sum_{k=1}^{\infty} \lambda(2^{-k}\tau x)\psi(2^{-k}\tau x)$$

and hence as in (49)

$$\begin{aligned}
\left(\psi(\tau\cdot)\check{}_j * f\right)(y) &= \left([\Lambda(\tau\cdot)\check{}_j * \Psi(\tau\cdot)\check{}_j] * [\psi(\tau\cdot)\check{}_j * f]\right)(y) \\
&\quad + \sum_{k=j+1}^{\infty} \left([\psi(\tau\cdot)\check{}_j * \lambda(\tau\cdot)\check{}_k)] * [\psi(\tau\cdot)\check{}_k * f]\right)(y)
\end{aligned} \tag{60}$$

for all $y \in \mathbb{R}^n$. As there one can obtain by lemma 2.3 and in view of $k \ge j$

$$\begin{aligned}
\left|\left(\psi(\tau\cdot)\check{}_j * \lambda(\tau\cdot)\check{}_k\right)(z)\right| &= \left|2^{jn}\tau^{-n}\left(\check{\psi} * \check{\lambda}_{k-j}\right)(2^j\tau^{-1}z)\right| \\
&= \tau^{-n}\left|2^{jn}\left(\check{\psi} * \lambda(2^{j-k}\cdot)\check{}\right)(2^j\tau^{-1}z)\right| \\
&\le C_{N,\psi,\lambda}\frac{2^{jn}\tau^{-n}2^{(j-k)N}}{(1+2^j\tau^{-1}|z|)^a} \le C'_{N,\psi,\lambda}\frac{2^{jn}2^{(j-k)N}}{(1+2^j|z|)^a}
\end{aligned}$$

for all $N \in \mathbb{N}$ and

$$\begin{aligned}|(\Psi(\tau\cdot)\check{}_j * \Lambda(\tau\cdot)\check{}_j(z))| &= 2^{2jn}\tau^{-2n}\int_{\mathbb{R}^n}\check{\Psi}(2^j\tau^{-1}(z-y))\check{\Lambda}(2^j\tau^{-1}y)\\ &= 2^{jn}\tau^{-n}\left(\check{\Psi}*\check{\Lambda}\right)(2^j\tau^{-1}z)\\ &\le C_{\Psi,\Lambda}\frac{2^{jn}\tau^{-n}}{(1+2^j\tau^{-1}|z|)^a} \le C'_{\Psi,\Lambda}\frac{2^{jn}}{(1+2^j|z|)^a},\end{aligned}$$

where $C'_{N,\psi,\lambda}$ and $C'_{\Psi,\Lambda}$ do not depend on $\tau \in [1,2]$. From (60) we get the analogous result of (51), namely

$$\begin{aligned}&(\psi^{*'}_{2^{-j}\tau}f)_a(x)\\ &\le C_N\sum_{k=j}^{\infty}2^{(j-k)N'}[(\psi^{*'}_{2^{-k}\tau}f)_a(x)]^{1-r}\int_{\mathbb{R}^n}\frac{2^{kn}\|\left(\psi(2^{-k}\tau\cdot)\check{}*f\right)(z)|E\|^r}{(1+2^k|x-z|)^{ar}}\,dz. \end{aligned}\tag{61}$$

Furthermore, we derive an estimate for Ψ. However, this will be of slightly different shape because we don't want to estimate the term with $\Psi(\tau\cdot)$, but with Ψ itself. We start with the analogue of (60) for $j=1$ and Ψ instead of $\psi(\tau\cdot)\check{}_j$

$$\left(\check{\Psi}*f\right)(y)=\left([\Lambda(\tau\cdot)\check{}*\Psi(\tau\cdot)\check{}]*\left[\check{\Psi}*f\right]\right)(y)+\sum_{k=1}^{\infty}\left(\left[\check{\Psi}*\lambda(\tau\cdot)\check{}_k\right)\right]*[\psi(\tau\cdot)\check{}_k*f]\right)(y)$$

for all $y \in \mathbb{R}^n$. Now by lemma 2.3 we have

$$\left|\left(\check{\Psi}*\lambda(\tau\cdot)\check{}_k\right)(z)\right| \le C_{\Psi,\lambda,N}\frac{\tau^N 2^{-kN}}{(1+|z|)^a} \le C'_{\Psi,\lambda,N}\frac{2^{-kN}}{(1+|z|)^a}$$

and obviously

$$|(\Lambda(\tau\cdot)\check{}*\Psi(\tau\cdot)\check{})(z)| \le C_{\Psi,\Lambda}\frac{1}{(1+|z|)^a}$$

with $C'_{\Psi,\lambda,N}$ and $C_{\Psi,\Lambda}$ independent of $\tau \in [1,2]$. Thereby we come with the same arguments as in the original proof to

$$\begin{aligned}(\Psi^*f)_a(x) &\le C_N[(\Psi^*f)_a(x)]^{1-r}\int_{\mathbb{R}^n}\frac{\|\left(\check{\Psi}*f\right)(z)|E\|^r}{(1+|x-z|)^{ar}}\,dz\\ &+C_N\sum_{k=1}^{\infty}2^{-kN'}[(\psi^{*'}_{2^{-k}\tau}f)_a(x)]^{1-r}\int_{\mathbb{R}^n}\frac{2^{kn}\|\left(\psi(2^{-k}\tau\cdot)\check{}*f\right)(z)|E\|^r}{(1+2^k|x-z|)^{ar}}\,dz.\end{aligned}\tag{62}$$

If $r \le 1$, we use lemma 2.5, applied to inequality (61) and (62), which are valid for all $N' \in \mathbb{N}$, with $d_j = (\psi^{*'}_{2^{-j}\tau}f)_a(x)$. Notice that the results about condition (52) can be transfered from step 3 of the proof of theorem 2.1 and hold for the d_j's here, too. Therefore, we obtain the analogue of (53) in the case $r \le 1$

$$(\psi^{*'}_{2^{-j}\tau}f)_a(x)^r \le C'_N\sum_{k=j}^{\infty}2^{(j-k)Nr}\int_{\mathbb{R}^n}\frac{2^{kn}\|\left(\psi(2^{-k}\tau\cdot)\check{}*f\right)(z)|E\|^r}{(1+2^k|x-z|)^{ar}}\,dz \tag{63}$$

and the analogue of (54)

$$(\Psi^* f)_a(x)^r \le C'_N \int_{\mathbb{R}^n} \frac{\| \left(\check{\Psi} * f\right)(z)|E\|^r}{(1+|x-z|)^{ar}}\, dz \\ + C'_N \sum_{k=1}^{\infty} 2^{-kNr} \int_{\mathbb{R}^n} \frac{2^{kn}\| \left(\psi(2^{-k}\tau\cdot)^\smile * f\right)(z)|E\|^r}{(1+2^k|x-z|)^{ar}}\, dz, \tag{64}$$

where the constant C'_N does not depend on $r \in (0,1]$, $f \in \mathcal{S}'(\mathbb{R}^n, E)$, $j \in \mathbb{N}$ and $\tau \in [1,2]$. As in the initial proof the assertion follows for $r > 1$ as well. We only used Hölder's inequality there.

In the $F^s_{p,q}(E)$-case we argue as follows: We raise to the power of $\frac{q}{r}$, integrate over $\tau \in [1,2]$ with respect to $\frac{d\tau}{\tau}$, take the $\frac{q}{r}$-th root and obtain

$$\left(\int_{2^{-j}}^{2^{-j+1}} \left((\psi_t^{*'} f)_a(x)\right)^q \frac{dt}{t}\right)^{\frac{r}{q}} \\ \le C'_N \left(\int_1^2 \left(\sum_{k=j}^{\infty} 2^{(j-k)Nr} \int_{\mathbb{R}^n} \frac{2^{kn}\| \left(\psi(2^{-k}\tau\cdot)^\smile * f\right)(z)|E\|^r}{(1+2^k|x-z|)^{ar}}\, dz\right)^{\frac{q}{r}} \frac{dt}{t}\right)^{\frac{r}{q}}$$

and

$$(\Psi^* f)_a(x)^r \le C''_N \int_{\mathbb{R}^n} \frac{\| \left(\check{\Psi} * f\right)(z)|E\|^r}{(1+|x-z|)^{ar}}\, dz \\ + C''_N \left(\int_1^2 \left(\sum_{k=1}^{\infty} 2^{-kNr} \int_{\mathbb{R}^n} \frac{2^{kn}\| \left(\psi(2^{-k}\tau\cdot)^\smile * f\right)(z)|E\|^r}{(1+2^k|x-z|)^{ar}}\, dz\right)^{\frac{q}{r}} \frac{dt}{t}\right)^{\frac{r}{q}}.$$

If $r \le q$, we can use the (generalized) Minkowski inequality two times and get

$$\left(\int_{2^{-j}}^{2^{-j+1}} \left((\psi_t^{*'} f)_a(x)\right)^q \frac{dt}{t}\right)^{\frac{r}{q}} \\ \le C'_N \sum_{k=j}^{\infty} 2^{(j-k)Nr} \int_{\mathbb{R}^n} \frac{2^{kn}}{(1+2^k|x-z|)^{ar}} \left(\int_{2^{-k}}^{2^{-k+1}} \| \left(\psi(t\cdot)^\smile * f\right)(z)|E\|^q \frac{dt}{t}\right)^{\frac{r}{q}} dz$$

and

$$(\Psi^* f)_a(x)^r \le C''_N \int_{\mathbb{R}^n} \frac{\| \left(\check{\Psi} * f\right)(z)|E\|^r}{(1+|x-z|)^{ar}}\, dz \\ + C''_N \sum_{k=1}^{\infty} 2^{-kNr} \int_{\mathbb{R}^n} \frac{2^{kn}}{(1+2^k|x-z|)^{ar}} \left(\int_{2^{-k}}^{2^{-k+1}} \| \left(\psi(t\cdot)^\smile * f\right)(z)|E\|^q \frac{dt}{t}\right)^{\frac{r}{q}} dz.$$

This yields the estimates (53) and (54), only with the terms

$$\left(\int_{2^{-k}}^{2^{-k+1}} \| \left(\psi(t\cdot)^\smile * f\right)(z)|E\|^q \frac{dt}{t}\right)^{\frac{1}{q}}$$

instead of $\| \left(\check{\psi}_k * f \right)(z)|E\|$ and

$$\left(\int_{2^{-j}}^{2^{-j+1}} \left((\psi_t^{*'} f)_a(x) \right)^q \frac{dt}{t} \right)^{\frac{1}{q}}$$

instead of $(\psi_j^* f)_a(x)$. Now we pick an a so large that we can choose r with $\frac{n}{a} < r < \min(p,q)$. Now we reconstruct the further steps in the initial proof with the given "replacements" and obtain immediately

$$\|(\Psi^* f)_a|L_p\| + \left\| \left(\sum_{j=1}^{\infty} 2^{jsq} \int_{2^{-j}}^{2^{-j+1}} \left((\psi_t^{*'} f)_a \right)^q \frac{dt}{t} \right)^{\frac{1}{q}} \Big| L_p \right\|$$
$$\leq C' \|(\Psi \hat{f})\check{}\,|L_p(E)\| + C' \left\| \left(\sum_{j=1}^{\infty} 2^{jsq} \int_{2^{-j}}^{2^{-j+1}} \|(\psi(t\cdot)\hat{f})\check{}\,|E\|^q \frac{dt}{t} \right)^{\frac{1}{q}} \Big| L_p \right\|,$$

which obviously gives

$$\|(\Psi^* f)_a|L_p\| + \left\| \left(\int_0^1 t^{-sq} \left((\psi_t^{*'} f)_a \right)^q \frac{dt}{t} \right)^{\frac{1}{q}} \Big| L_p \right\|$$
$$\leq C'' \|(\Psi \hat{f})\check{}\,|L_p(E)\| + C'' \left\| \left(\int_0^1 t^{-sq} \|(\psi(t\cdot)\hat{f})\check{}\,|E\|^q \frac{dt}{t} \right)^{\frac{1}{q}} \Big| L_p \right\|.$$

In the $B^s_{p,q}(E)$-case we start with (63) and arrive at

$$2^{jsr} (\psi^{*'}_{2^{-j}\tau} f)_a(x)^r \leq C \sum_{k=j}^{\infty} 2^{(j-k)\delta} 2^{ksr} M \left(\| \left(\psi(2^{-k}\tau\cdot)\check{} * f \right) |E\|^r \right)(x)$$

as in the original proof and at an analogous result for $(\Psi^* f)_a(x)^r$. Now we take the $L_{\frac{p}{r}}$-norm, use the Minkowski inequality and the boundedness of the maximal operator from $L_{\frac{p}{r}}$ to $L_{\frac{p}{r}}$ and come to

$$2^{jsr} \|(\psi^{*'}_{2^{-j}\tau} f)_a(x)|L_p\|^r \leq C' \sum_{k=j}^{\infty} 2^{(j-k)\delta} 2^{ksr} \left\| \left(\psi(2^{-k}\tau\cdot)\check{} * f \right) |L_p(E) \right\|^r$$

and to an analogous result for $(\Psi^* f)_a(x)^r$. Now we integrate over $\tau \in [1,2]$ with respect to $\frac{d\tau}{\tau}$, argue as in the $F^s_{p,q}(E)$-case and use a suitable estimate for the l_q-norm as in lemma 2.4. Then we obtain the desired result for $B^s_{p,q}(E)$.

The characterizations (57) and (58) hold true by the same arguments as in step 4 of the proof of theorem 2.1. One just has to notice that lemma 2.5 is applied to $d_j = (\psi^{*'}_{2^{-j}\tau} f)_a(x)$ instead of $(\psi_j^* f)_a(x)$ which makes no big difference for the calculations. □

2.2 Explicit norms and characterizations

Below we will take a look at some examples of equivalent norms and characterizations following from theorem 2.1 which will be of use later on.

Example 2.8. Let $k_0, k^0 \in \mathcal{S}(\mathbb{R}^n)$, $\hat{k_0}(0) \neq 0$, $\hat{k^0}(0) \neq 0$. Let $N \in \mathbb{N}_0$ with $2N > s$. Then the functions $\Psi := \hat{k_0}$ and $\psi := \widehat{\Delta^N k^0}$ fulfil the conditions (23) and (24) for a suitable small $\varepsilon > 0$ and (25) with $S = 2N-1$ since $\widehat{\Delta k^0}(x) = -|x|^2 \hat{k^0}(x)$. In particular, k_0 and k^0 can be chosen such that *supp* k_0, *supp* $k^0 \subset B$. This is where the expression "local means" comes from because if $f \in \mathcal{S}'(\mathbb{R}^n, E)$ is e.g. a regular distribution, then we have by a variable substitution

$$(\hat{k_0}(2^{-j}\cdot)\hat{f})\check{}(x) = 2^{jn}\left(k_0(2^j\cdot) * f\right)(x) = \int_B k_0(y) f(x - 2^{-j}y)\, dy$$

and an analogue for $\Delta^N k^0$ so that for a calculation only the values of f in the small domain $\{y \in \mathbb{R}^n : |y| \leq 2^{-j}\}$ are necessary to know . If we set $k^N := \Delta^N k^0$, it follows

Proposition 2.9. *Let $2N > s$.*
(i) Let $0 < p \leq \infty$ and $0 < q \leq \infty$. Then

$$\|f|B^s_{p,q}(E)\|_{k_0,k^N} := \|k_0 * f|L_p(E)\| + \left(\sum_{j=1}^{\infty} 2^{jsq}\|k_j^N * f|L_p(E)\|^q\right)^{\frac{1}{q}}$$

(modified for $q=\infty$) is an equivalent norm for $\|\cdot|B^s_{p,q}(E)\|$. It holds

$$B^s_{p,q}(E) = \left\{f \in \mathcal{S}'(\mathbb{R}^n, E) : \|f|B^s_{p,q}(E)\|_{k_0,k^N} < \infty\right\}.$$

(ii) Let $0 < p < \infty$ and $0 < q \leq \infty$. Then

$$\|f|F^s_{p,q}(E)\|_{k_0,k^N} := \|k_0 * f|L_p(E)\| + \left\|\left(\sum_{j=1}^{\infty} 2^{jsq}\|k_j^N * f|E\|^q\right)^{\frac{1}{q}} \Big| L_p\right\|$$

(modified for $q=\infty$) is an equivalent norm for $\|\cdot|F^s_{p,q}(E)\|$. It holds

$$F^s_{p,q}(E) = \left\{f \in \mathcal{S}'(\mathbb{R}^n, E) : \|f|F^s_{p,q}(E)\|_{k_0,k^N} < \infty\right\}.$$

Example 2.10. In our remarks we follow [Tri97], section 12.2, p. 59. Let $h(x) := (1+|x|^2)^{-\frac{n+1}{2}}$. By using [StW90], theorem 1.14, p. 6, we obtain

$$\widehat{e^{-t|\cdot|}}(x) = d_n h_{1/t}(x) = d_n t^{-n} \frac{1}{(1+|\frac{x}{t}|^2)^{\frac{n+1}{2}}} = d_n \frac{t}{(t^2+|x|^2)^{\frac{n+1}{2}}} \tag{65}$$

for a suitable constant $d_n > 0$. The function $P(x,t) = d_n(t^2+|x|^2)^{\frac{n+1}{2}}$ and also its partial derivatives with respect to t are harmonic in the domain $\{(x,t) : x \in \mathbb{R}^n, t > 0\}$ for instance by basic properties of the Fourier transform.

Let $f \in \mathcal{S}(\mathbb{R}^n, E)$. Then

$$u(x,t) := (e^{-t|\cdot|}\hat{f})^{\check{}}(x) = d_n \left(f * \frac{t}{(|\cdot|^2 + t^2)^{\frac{n+1}{2}}} \right)(x)$$

is harmonic as well and hence its partial derivatives with respect to t for which

$$\frac{\partial^k u(x,t)}{\partial t^k} = ((-1)^k |\cdot|^k e^{-t|\cdot|}\hat{f})^{\check{}}(x)$$

holds.

We choose a $\phi \in \mathcal{S}(\mathbb{R}^n)$ with $\phi(x) = 1$ for $|x| \le 1$ and $\phi(x) = 0$ for $|x| > \frac{3}{2}$ and set $\Psi := \phi$ and $\psi^k(\xi) := |\xi|^k e^{-\xi}$. It follows

$$(\psi^k(t\cdot)\hat{f})^{\check{}}(x) = (-1)^k t^k \frac{\partial^k u(x,t)}{\partial t^k}$$

at least for $f \in \mathcal{S}(\mathbb{R}^n, E)$. The functions Ψ und ψ^k fulfil the support conditions (23) and (24) but ψ^k is not arbitrarily often differentiable. But, for instance, there exist all partial derivatives of ψ^{2k} up to the order $2k$. The moment conditions (25) are fulfilled for all derivatives up to the order $2k-1$.

This means that we cannot apply theorem 2.1 including the subsequent proposition 2.7 directly for this functions. Nevertheless, we will try to obtain the desired result in this case as well. For this we can and will choose $k \ge k_0$ in a suitable dependency of p, q and s.

Let $f \in B^s_{p,q}(E)$ resp. $F^s_{p,q}(E)$ be given. In consequence of $|\cdot|^k e^{-t|\cdot|} \notin \mathcal{S}(\mathbb{R}^n)$ we cannot use the initial definition of $(e^{-t|\cdot|}\hat{f})^{\check{}}$. So we decompose f into $f_1 := (\phi\hat{f})^{\check{}}$ and $f_2 := ((1-\phi)\hat{f})^{\check{}}$.[6] By the assumptions $f_1 \in L_p^{B_2}(E)$ and therefore by Nikolskii's inequality (see (10)) $f_1 \in L_\infty$. Hence

$$u_1(x,t) = d_n \left(f_1 * \frac{t}{(|\cdot|^2 + t^2)^{\frac{n+1}{2}}} \right)(x)$$

is well-defined, even bounded (also in t) as a convolution of an L_1-function with an $L_\infty(E)$-function. The function is harmonic in the domain $\{(x,t) : x \in \mathbb{R}^n, t > 0\}$. Because $h(x)$ is arbitrarily often differentiable and its derivatives are integrable, $u_1(x,t)$ is arbitrarily often differentiable with respect to x. The differentiability with respect to t follows from $u_1(x,t) = d_n \left(h_{1/t} * f_1\right)(x)$. So the (classical) partial derivatives with respect to t exist and are harmonic, too.

The functions $[1 + |\cdot|^2]^\sigma e^{-t|\cdot|}(1-\Phi)$, whose $\mathcal{S}(\mathbb{R}^n)$-seminorms for $t > \delta$ are uniformly bounded, are Fourier multipliers for $B^s_{p,q}(E)$ resp. $F^s_{p,q}(E)$ (see (16)) for all $\sigma \in \mathbb{R}$ since $e^{-t|\cdot|}(1-\phi) \in \mathcal{S}(\mathbb{R}^n)$. But then it follows from the lift property of these spaces (see (17)) and the Sobolev embeddings (18) and (19) that

$$u_2(\cdot,t) = \left([1 + |\cdot|^2]^\sigma e^{-t|\cdot|}(1-\phi) \left([1 + |\xi|^2]^{-\sigma}\hat{f} \right) \right)^\vee \in B^s_{\infty,\infty}$$

[6] If $f \in B^s_{p,q}(E)$ resp. $F^s_{p,q}(E)$, then f_2 is well-defined by the Fourier multiplier theorem (16) with $f_2 \in B^s_{p,q}(E)$ resp. $F^s_{p,q}(E)$.

for all $s \in \mathbb{R}$. So $u_2(\cdot, t)$ is arbitrarily often differentiable with respect to x and bounded in the domain $\{x \in \mathbb{R}^n, t > \delta\}$ for a fixed $\delta > 0$ (by (15)). The differentiability relative to t is obvious. The function is harmonic which can be seen by easy calculations under using basic properties of the Fourier transform.

Analogous assertions hold true for $|\cdot|^k e^{-t|\cdot|}$ instead of $e^{-t|\cdot|}$ and therefore for the partial derivatives of $u_2(x,t)$. So $u(x,t) := u_1(x,t) + u_2(x,t)$ is well-defined for all $f \in B^s_{p,q}(E)$ resp. $F^s_{p,q}(E)$ and for arbitrary $s \in \mathbb{R}$, $0 < p \leq \infty$ (resp. $< \infty$) and $0 < q \leq \infty$, arbitrarily often differentiable, harmonic in the domain $\{x \in \mathbb{R}^n, t > 0\}$ and bounded on $\{x \in \mathbb{R}^n, t > \delta\}$ for fixed $\delta > 0$. An analogue is valid for the partial derivatives.

Now we have the necessary tools together to formulate and proof the desired result. We will show how to modify the initial proof to incorporate the functions above.

Proposition 2.11. *Let $d > 0$ and $s \in \mathbb{R}$.*

(i) Let $0 < p \leq \infty$, $0 < q \leq \infty$. Then there exists a k_0 such that for all $k \geq k_0$

$$\|f|B^s_{p,q}(E)\|_{\phi,k} := \|(\phi\hat{f})\check{}|L_p(E)\| + \left(\int_0^1 t^{(k-s)q}\|\frac{\partial^k u(\cdot,t)}{\partial t^k}|L_p(E)\|^q \frac{dt}{t}\right)^{\frac{1}{q}}$$

and

$$\|f|B^s_{p,q}(E)\|^{\sup}_{\phi,k} := \left\|\sup_{|\cdot-y|\leq d} \|(\phi\hat{f})\check{}(y)|E\| \Big| L_p\right\|$$
$$+ \left(\int_0^1 t^{(k-s)q} \left\|\sup \|\frac{\partial^k u(y,\tau)}{\partial t^k}|E\| \Big| L_p\right\|^q \frac{dt}{t}\right)^{\frac{1}{q}}$$

are equivalent norms for $\|\cdot|B^s_{p,q}(E)\|$, where sup *is the supremum for a fixed $x \in \mathbb{R}^n$ over $\{|x-y| \leq dt, t \leq \tau \leq 2t\}$. It holds*

$$B^s_{p,q}(E) = \left\{f \in \mathscr{C}^{-\infty}(E) : \|f|B^s_{p,q}(E)\|_{\phi,k} < \infty\right\}$$

and

$$B^s_{p,q}(E) = \left\{f \in \mathscr{C}^{-\infty}(E) : \|f|B^s_{p,q}(E)\|^{\sup}_{\phi,k} < \infty\right\}.$$

(ii) Let $0 < p < \infty$, $0 < q \leq \infty$. Then there exists a k_0 such that for all $k \geq k_0$

$$\|f|F^s_{p,q}(E)\|_{\phi,k} := \left\|(\phi\hat{f})\check{}|L_p(E)\right\| + \left\|\left(\int_0^1 t^{(k-s)q}\|\frac{\partial^k u(\cdot,t)}{\partial t^k}|E\|^q \frac{dt}{t}\right)^{\frac{1}{q}} \Big| L_p\right\|$$

and

$$\|f|F^s_{p,q}(E)\|^{\sup}_{\phi,k} := \left\|\sup_{|\cdot-y|\leq d} \|(\phi\hat{f})\check{}(y)|E\| \Big| L_p\right\|$$
$$+ \left\|\left(\int_0^1 t^{-sq} \sup \|\frac{\partial^k u(y,\tau)}{\partial t^k}|E\|^q\right)^{\frac{1}{q}} \Big| L_p\right\|$$

are equivalent norms for $\|\cdot|F^s_{p,q}(E)\|$, *where* sup *is the supremum for a fixed* $x \in \mathbb{R}^n$ *over* $\{|x-y| \le dt, t \le \tau \le 2t\}$. *It holds*

$$F^s_{p,q}(E) = \left\{f \in \mathscr{C}^{-\infty}(E) : \|f|F^s_{p,q}(E)\|_{\phi,k} < \infty\right\} \tag{66}$$

and

$$F^s_{p,q}(E) = \left\{f \in \mathscr{C}^{-\infty}(E) : \|f|F^s_{p,q}(E)\|^{\sup}_{\phi,k} < \infty\right\}. \tag{67}$$

Proof. We like to recall the relation

$$(-1)^k t^k \frac{\partial^k u(x,t)}{\partial t^k} = \left((t|\cdot|)^k e^{-t|\cdot|}\hat{f}\right)^\vee(x).$$

We put $\Psi := \phi$ and $\psi := |\cdot|^k e^{-|\cdot|}$, where we will choose k so large (afterwards) that the steps of the proof of theorem 2.1 resp. proposition 2.7 can be applied to the function ψ, which is not arbitrarily often differentiable, as well.

Let Φ, $\varphi \in \mathcal{S}(\mathbb{R}^n)$ with (31) be given. Then we can choose $\Lambda, \lambda \in \mathcal{S}(\mathbb{R}^n)$ with (34) and (35). We obtain for $f \in B^s_{p,q}(E)$ resp. $F^s_{p,q}(E)$

$$f = \left(\check{\Lambda} * \check{\Phi}\right) * f + \sum_{k=1}^\infty \left(\check{\lambda}_k * \check{\varphi}_k\right) * f,$$

where the equality is to be understood in the sense of $\mathcal{S}'(\mathbb{R}^n, E)$-convergence. Because $(\tau|\cdot|)^k e^{-\tau|\cdot|}(1-\phi) \in \mathcal{S}(\mathbb{R}^n)$, it follows

$$\begin{aligned} \left((\psi(\tau\cdot)(1-\phi))\check{} * f\right)(y) &= \left(\left(\check{\Lambda} * \check{\Phi}\right) * \left((\psi(\tau\cdot)(1-\phi))\check{} * f\right)\right)(y) \\ &\quad + \sum_{k=1}^\infty \left(\left(\check{\lambda}_k * \check{\varphi}_k\right) * \left((\psi(\tau\cdot)(1-\phi))\check{} * f\right)\right)(y) \end{aligned} \tag{68}$$

for all $y \in \mathbb{R}^n$ and

$$\begin{aligned} \check{\phi} * f &= (\check{\Lambda} * \check{\Phi}) * (\check{\phi} * f) + \sum_{k=1}^\infty (\check{\lambda}_k * \check{\varphi}_k) * (\check{\phi} * f) \\ &= (\check{\Lambda} * \check{\Phi}) * (\check{\phi} * f) + \sum_{k=1}^K (\check{\lambda}_k * \check{\varphi}_k) * (\check{\phi} * f) \end{aligned}$$

in L_∞ because *supp* $\lambda(2^{-k}\cdot) \cap$ *supp* $\phi = \emptyset$ for all $k > K$ for a certain $K \in \mathbb{N}$ (see (34)), $\check{\phi} * f \in L_\infty(E)$ and $\check{\lambda}_k * \check{\varphi}_k \in L_1$. Hence it follows by basic properties of the convolution of functions from $L_\infty(E)$ with functions from L_1

$$\begin{aligned} &\left(\psi(\tau\cdot)\check{} * \check{\phi} * f\right) = \left((\psi(\tau\cdot)\phi)\check{} * f\right) \\ &= \left(\check{\Lambda} * \check{\Phi}\right) * \left((\psi(\tau\cdot)\phi)\check{} * f\right) + \sum_{k=1}^\infty \left(\check{\lambda}_k * \check{\varphi}_k\right) * \left((\psi(\tau\cdot)\phi)\check{} * f\right) \end{aligned} \tag{69}$$

in $L_\infty(E)$. From the relations (68) and (69) we obtain the desired intermediate result

$$\left(\psi(\tau\cdot)\check{} * f\right)(y) = \left(\left(\check{\Lambda} * \check{\Phi}\right) * \left(\psi(\tau\cdot)\check{} * f\right)\right)(y) + \sum_{k=1}^\infty \left(\left(\check{\lambda}_k * \check{\varphi}_k\right) * \left(\psi(\tau\cdot)\check{} * f\right)\right)(y)$$

for all $y \in \mathbb{R}^n$. Now it remains to estimate the integrals $I_{j,k}$ in (37) with the ψ used here. For this we applied lemma 2.3 in the first step of the proof, where we assumed $\psi \in \mathcal{S}(\mathbb{R}^n)$. If we take a deeper look at the proof and the subsequent steps, we see that it suffices that for given ψ and for given a there exist bounded derivatives up to the order $\lceil a \rceil + n + 2$ such that

$$\| \left(1 + |\cdot|^{\lceil 2a-s \rceil +1}\right) D^\alpha \psi | L_1 \| < \infty \text{ for } |\alpha| \le \lceil a \rceil + n + 2$$

and that

$$D^\alpha \psi(0) = 0 \text{ for } |\alpha| \le \lfloor s \rfloor .$$

But these requirements are fulfilled by our function $\psi(\xi) = |\xi|^k e^{-|x|}$if we just take k larger oder equal to a suitable k_1.

The remaining argument from the first step of the proof of theorem 2.1 resp. proposition 2.7 can be carried over directly. At that it suffices to show the assertion for a special $a > \frac{n}{p}$ resp. $a > \frac{n}{\min(p,q)}$ as mentioned at the end of the proof of 2.7. Hence one can easily derive how large k_1 has to be chosen.

In the second part we again change the roles of Φ and Ψ resp. φ and ψ as in the proof of assertion 2.7. We choose for Ψ and $\psi(\tau\cdot)$ functions $\Lambda^\tau, \lambda^\tau \in \mathcal{S}(\mathbb{R}^n)$ with (34) and (35). This is possible because we don't need that $\psi(\tau\cdot)$ is differentiable at 0 (see the construction in lemma 2.2). Thus we obtain (in the special case $\tau = 1$)

$$1 = \Lambda(x)\Psi(x) + \sum_{j=1}^{\infty} \lambda(2^{-j}x)\psi(2^{-j}x).$$

If we use that $0 \notin supp\ \lambda(2^{-j}x)\psi(2^{-j}x)$ and that these functions are arbitrarily often differentiable, it follows (using the example of $\tau = 1$)

$$\hat{f} = \Lambda\Psi\hat{f} + \sum_{j=1}^{\infty} \lambda(2^{-j}\cdot)\psi(2^{-j}\cdot)\hat{f}$$

in $\mathcal{S}'(\mathbb{R}^n, E)$ now. We multiplicate the equality by ϕ resp. $(1-\phi)$, apply the Fourier transform and come for $i = 1, 2$ to

$$f_i = \check{\Lambda} * \left(\check{\Psi} * f_i\right) + \sum_{k=1}^{\infty} \check{\lambda}_k * \left(\check{\psi}_k * f_i\right)$$

with $f_1 = (\phi\hat{f})\check{}$ and $f_2 = ((1-\phi)\hat{f})\check{}$ and convergence in L_∞ (the sum is finite again) resp. $\mathcal{S}'(\mathbb{R}^n, E)$. After adding both equalities for $i = 1, 2$ and by using the definition of the convolution with $\check{\psi}_k$ it follows

$$f = \check{\Lambda} * \left(\check{\Psi} * f\right) + \sum_{k=1}^{\infty} \check{\lambda}_k * \left(\check{\psi}_k * f\right)$$

in $\mathcal{S}'(\mathbb{R}^n, E)$, where $\check{\psi}_k * f$ is a bounded, arbitrarily often differentiable function by the preliminary considerations. We convolute with $\check{\varphi}_j$, use the associativity of the convolution and obtain (in case $\tau = 1$)

$$\left(\check{\varphi}_j * f\right)(y) = \left((\check{\varphi}_j * \check{\Lambda}) * (\check{\Psi} * f)\right)(y) + \sum_{k=1}^{\infty} \left((\check{\varphi}_j * \check{\lambda}_k) * (\check{\psi}_k * f)\right)(y)$$

for all $y \in \mathbb{R}^n$. Now the integrals $I_{j,k}$ are the same as in the proof of proposition 2.7 and can be estimated correspondingly so that we can complete this step and get the analogue of (59).

Now we modify step 2 of the proof for Ψ and ψ. We again choose Λ, λ for Ψ, ψ by lemma 2.2. This yields (in the case $\tau = 1$)

$$\check{f} = \Lambda_j \Psi_j \hat{f} + \sum_{k=j+1}^{\infty} \lambda(2^{-k}\cdot)\psi(2^{-k}\cdot)\hat{f}$$

in $\mathcal{S}'(\mathbb{R}^n, E)$. We apply the Fourier transform and obtain with $(1-\phi)\psi(2^{-j}\cdot) \in \mathcal{S}(\mathbb{R}^n)$

$$\begin{aligned}\left(\left(\psi(2^{-j}\cdot)(1-\phi)\right)\check{} * f\right)(y) &= \left(\left(\check{\Lambda}_j * \check{\Psi}_j\right) * \left(\left(\psi(2^{-j}\cdot)(1-\phi)\right)\check{} * f\right)\right)(y) \\ &\quad + \sum_{k=j+1}^{\infty} \left(\left(\check{\lambda}_k * \check{\psi}_k\right) * \left(\left(\psi(2^{-j}\cdot)(1-\phi)\right)\check{} * f\right)\right)(y)\end{aligned}$$

for all $y \in \mathbb{R}^n$. Analogously as in (69) we arrive at

$$\begin{aligned}\left(\psi(2^{-j}\cdot)\phi\right)\check{} * f = \left(\check{\Lambda}_j * \check{\Psi}_j\right) * \left(\left(\psi(2^{-j}\cdot)\phi\right)\check{} * f\right) + \\ \sum_{k=j+1}^{\infty} \left(\check{\lambda}_k * \check{\psi}_k\right) * \left(\left(\psi(2^{-j}\cdot)\phi\right)\check{} * f\right)\end{aligned}$$

in $L_\infty(E)$. By adding this two equalities we finally derive (for the case $\tau = 1$)

$$\left(\check{\psi}_j * f\right)(y) = \left(\left(\check{\Lambda}_j * \check{\Phi}_j\right) * \left(\check{\psi}_j * f\right)\right)(y) + \sum_{k=j+1}^{\infty} \left(\left(\check{\lambda}_k * \check{\psi}_k\right) * \left(\check{\psi}_j * f\right)\right)(y)$$

for all $y \in \mathbb{R}^n$. This is the analogon for (60) (for the case $\tau = 1$).

The subsequent estimates are based on lemma 2.3. If we consider its proof (as done once above) and the next steps, we see that it suffices that for given ψ and a there exist derivatives up to the order $\max(a+n+1, a+1) = a+n+1$ (case $r \le 1$ resp. $r > 1$ in the proofs) and that for all $|\alpha| \le a+n+1$ and all $N \in \mathbb{N}_0$

$$\|(1+|\cdot|^N)D^\alpha \psi | L_1\|$$

is finite. But our function ψ with $\psi(\xi) = |\xi|^k e^{-|x|}$ fulfils these requirements if k is chosen larger as a given k_2. Therefore we obtain the analogon of (61). The estimate (62) for $\Psi * f$ follows as well because Ψ satisfies all conditions of the theorem resp. the proposition. So the analogues of (63) and (64) in the case $r \le 1$ can be derived from lemma 2.5. For that we have to show the condition (52) from lemma 2.5 for our ψ and for a $N_0 \in \mathbb{N}$. But this follows if $f \in B^s_{p,q}(E)$ or $F^s_{p,q}(E)$ for any $s \in \mathbb{R}$, $0 < p \le \infty$ (resp. $< \infty$) and $0 < q \le \infty$, i.e. if $f \in \mathscr{C}^{-\infty}(E)$ (see (20)) is assumed: Then we have

$$d_j = (\psi_j^* f)_a(x) \le \left(((1-\phi)\psi)_j^* f\right)_a(x) + ((\phi\psi)_j^* f)_a(x).$$

The first summand can be estimated as in step 3 or 4 of the proof of theorem 2.1 because $(1-\phi)\psi \in \mathcal{S}(\mathbb{R}^n)$. For the second summand we have

$$\|(\phi\psi)\check{}_j * f|L_\infty(E)\| \leq \|\check{\phi}_j * f|L_\infty\| \cdot \|\check{\psi}_j|L_1\| \leq c\|\check{\phi}_j * f|L_\infty(E)\|.$$

We choose a smooth dyadic resolution of unity consisting of the functions $\Phi = \varphi_0$ and $\{\varphi_j\}_{j\in\mathbb{N}}$ with $\varphi_j(\cdot) = \varphi_1(2^{-j+1}\cdot)$ for $j = 2, \ldots$ (see (14)) and obtain by Nikolskii's inequality (see (10))

$$\begin{aligned}
\|\check{\phi}_j * f|L_\infty(E)\| &\leq \sum_{k=0}^{j+1} \|\check{\phi}_j * (\varphi_k \hat{f})\check{}|L_\infty(E)\| \leq c \sum_{k=0}^{j+1} \|(\varphi_k \hat{f})\check{}|L_\infty(E)\| \\
&\leq c' \sum_{k=0}^{j+1} 2^{\frac{kn}{p}} \|(\varphi_k \hat{f})\check{}|L_p(E)\| \leq c'' \left(\sum_{k=0}^{j+1} 2^{k\left(\frac{n}{p}+\varepsilon\right)q} \|(\varphi_k \hat{f})\check{}|L_p(E)\|^q \right)^{\frac{1}{q}} \\
&\leq c' \max\left(2^{j\left(\frac{n}{p}+\varepsilon-s\right)}, 1\right) ||f|B^s_{p,q}(E)||.
\end{aligned}$$

So the desired condition (52) with $N_0 \geq \lceil \frac{n}{p} + \varepsilon - s \rceil$ follows. The remaining steps in the proof of proposition 2.7 can be transfered directly without any changes. It suffices again to proove the assertion for an $a > \frac{n}{\min(p,q)}$.

Because we assumed $f \in \mathscr{C}^{-\infty}(E)$ for defining the convolution with $e^{-|\cdot|}$ a priori, we obtain the best possible results with the characterizations (66) and (67). □

Remark 2.12. We follow [Tri97], theorem 12.5 (i) and (ii), p. 64. We want to replace ϕ in our proposition 2.11 by the function $e^{-|\cdot|}$ so that $(\Psi\hat{f})\check{}$ is harmonic as well. But this will only work for $p > \frac{n}{n+1}$. Namely in this case m with $m(\xi) = e^{-|\xi|}$ is a Fourier multiplier for $L_p^B(E)$, i.e. that there exists a $C > 0$ such that for all $f \in L_p^B(E)$ it is valid that

$$\|(e^{-|\cdot|}\hat{f})\check{}|L_p(E)\| \leq C\|f|L_p(E)\|.$$

Let's justify this: Let $0 < p < 1$ and $\lambda > n\left(\frac{1}{p} - 1\right)$. Then it follows from (12) that there exists a constant $c > 0$ such that for all $m : \mathbb{R}^n \to \mathbb{C}$ with $\check{m} \in L_1$ and for all $f \in L_p^B(E)$ the following holds

$$\begin{aligned}
\|(m\hat{f})\check{}|L_p(E)\| &= \|(m\phi\hat{f})\check{}|L_p(E)\| \\
&\leq \|\,(m\phi)\,\check{}|L_p\| \cdot \|f|L_p(E)\| \\
&\leq c\|(1+|\cdot|^2)^{\frac{\lambda}{2}}\,(m\phi)\,\check{}|L_1\| \cdot \|f|L_p(E)\| \\
&\leq c\|(1+|\cdot|^2)^{\frac{\lambda}{2}}\check{m}|L_1\| \cdot \|f|L_p(E)\|.
\end{aligned}$$

If $p \geq 1$, then we have by (11)

$$\|(m\hat{f})\check{}|L_p(E)\| \leq \|\check{m}|L_1\| \cdot \|f|L_p(E)\|.$$

The terms on the right-hand side are precisely finite if and only if $p > \frac{n}{n+1}$, see (65).

Using

$$\big(|\cdot|^k e^{-|\cdot|}\big)\,\hat{}\, = (-1)^k d_n \cdot \frac{\partial^k h_{1/t}(x,1)}{\partial t^k}$$

and the structure of the function h (see (65)) it follows by the same arguments that the functions $|\cdot|^k e^{-|\cdot|}$ are Fourier multipliers for $L_p^B(E)$ with $p > \frac{n}{n+1}$ as well.

Proposition 2.13. *Let $d > 0$ and $s \in \mathbb{R}$*

(i) Let $\frac{n}{n+1} < p \leq \infty$, $0 < q \leq \infty$ and $f \in B^s_{p,q}(E)$. Then there exist a $k_0 \in \mathbb{N}$ and a $c > 0$ such that for all $k \in \mathbb{N}$ with $k \geq k_0$

$$\|u(\cdot,1)|L_p(E)\| + \left(\int_0^1 t^{(k-s)q} \|\frac{\partial^k u(\cdot,t)}{\partial t^k}|L_p(E)\|^q \,\frac{dt}{t}\right)^{\frac{1}{q}}$$

and

$$\sum_{l=0}^{k-1} \left\| \sup_{|\cdot-y|\leq d} \|\frac{\partial^l u(y,1)}{\partial t^l}|E\| \;\Big|L_p \right\| + \left(\int_0^1 t^{(k-s)q} \left\|\sup \|\frac{\partial^k u(y,\tau)}{\partial t^k}|E\|\Big|L_p\right\|^q \frac{dt}{t}\right)^{\frac{1}{q}}$$

are bounded from above by $c\|\cdot|B^s_{p,q}(E)\|$ where sup *is the supremum for a fixed $x \in \mathbb{R}^n$ over $\{|x-y| \leq dt, t \leq \tau \leq 2t\}$.*

(ii) Let $\frac{n}{n+1} < p < \infty$, $0 < q \leq \infty$ and let $f \in F^s_{p,q}(E)$. Then there exist a $k_0 \in \mathbb{N}$ and a $c > 0$ such that for all $k \in \mathbb{N}$ with $k \geq k_0$

$$\|u(\cdot,1)|L_p(E)\| + \left\| \left(\int_0^1 t^{(k-s)q} \|\frac{\partial^k u(\cdot,t)}{\partial t^k}|E\|^q \,\frac{dt}{t}\right)^{\frac{1}{q}} \Big|L_p \right\|$$

and

$$\sum_{l=0}^{k-1} \left\| \sup_{|\cdot-y|\leq d} \|\frac{\partial^l u(y,1)}{\partial t^l}|E\| \;\Big|L_p \right\| + \left\| \left(\int_0^1 t^{(k-s)q} \sup \|\frac{\partial^k u(y,\tau)}{\partial t^k}|E\|^q\right)^{\frac{1}{q}} \Big|L_p \right\| \tag{70}$$

are bounded from above by $c\|\cdot|F^s_{p,q}(E)\|$ where sup *is the supremum for a fixed $x \in \mathbb{R}^n$ over $\{|x-y| \leq dt, t \leq \tau \leq 2t\}$.*

Proof. If we take a look at proposition 2.11 proven before, it suffices to estimate the first term of (70) by $||f|F^s_{p,q}(E)||$ (in the $F^s_{p,q}(E)$-case).

Let ϕ be chosen as in the preliminary thoughts of proposition 2.11 and $\Psi(\xi) := \phi(\xi)|\xi|^l e^{-|\xi|}$. We obtain

$$\begin{aligned}
&\sup_{|x-y|\leq d} \|\frac{\partial^l u(y,1)}{\partial t^l}|E\| \\
&\quad\leq \sup_{|x-y|\leq d} \|(\phi|\cdot|^l e^{-|\cdot|}\hat{f})\check{}\,(y)|E\| + \sup_{|x-y|\leq d} \|((1-\phi)|\cdot|^l e^{-|\cdot|}\hat{f})\check{}\,(y)|E\| \qquad (71)\\
&\quad\leq c(\Psi^* f)_a(x) + c \sup_{|x-y|\leq d} \|((1-\phi)e^{-|\cdot|}\hat{f})\check{}\,(y)|E\|
\end{aligned}$$

for all $a > 0$, where c depends on a. By (13) it follows if we choose $a > \frac{n}{p}$

$$\|(\Psi^* f)_a(x)|L_p\| \le c'\|(\Psi\hat{f})\check{}|L_p(E)\|.$$

Now we use that $|\cdot|^l e^{-|\cdot|}$ are Fourier multipliers for $L_p^B(E)$ with $p > \frac{n}{n+1}$ by the remark before and obtain

$$\|(\Psi^* f)_a(x)|L_p\| \le c''\|(\phi\hat{f})\check{}|L_p(E)\|. \tag{72}$$

Furthermore, let $g \in B_{p,1}^{\frac{n}{p}+\varepsilon}(E)$ for an $\varepsilon > 0$. If we choose $\{\varphi_k\}_{k=0}^{\infty}$ as a smooth dyadic resolution of unity (see (14)), then it is valid that for all $1 \le p \le \infty$

$$\|\sup_{|\cdot-y|\le d}\|g|E\||L_p\| \le \sum_{k=0}^{\infty}\|\sup_{|\cdot-y|\le d}\|(\varphi_k\hat{g})\check{}(y)|E\||L_p\|$$

and that for all $0 < p < 1$

$$\begin{aligned}\|\sup_{|\cdot-y|\le d}\|g|E\||L_p\| &\le \left(\sum_{k=0}^{\infty}\|\sup_{|\cdot-y|\le d}\|(\varphi_k\hat{g})\check{}(y)|E\||L_p\|^p\right)^{\frac{1}{p}}\\ &\le c\sup_{k\in\mathbb{N}_0}2^{k\varepsilon}\|\sup_{|\cdot-y|\le d}\|(\varphi_k\hat{g})\check{}(y)|E\||L_p\|\\ &\le c\sum_{k=0}^{\infty}2^{k\varepsilon}\|\sup_{|\cdot-y|\le d}\|(\varphi_k\hat{g})\check{}(y)|E\||L_p\|.\end{aligned}$$

By using that

$$\begin{aligned}\|\sup_{|\cdot-y|\le d}\|(\varphi_k\hat{g})\check{}(y)|E\||L_p\| &\le c'2^{ka}\left\|\sup_{y\in\mathbb{R}^n}\|\frac{(\varphi_k\hat{g})\check{}(\cdot-y)}{(1+|2^k y|)^a}|E\|\big|L_p\right\|\\ &= c'2^{ka}\|(\varphi_k^* g)_a|L_p(E)\|\end{aligned}$$

with c' independent of k we get by theorem 2.1 (or (13)) that

$$\|\sup_{|\cdot-y|\le d}\|g|E\||L_p\| \le c''\sum_{k=0}^{\infty}2^{k\varepsilon}2^{ka}\|(\varphi_k^* g)_a\big|L_p(E)\| \le c'''\|g|B_{p,1}^{a+\varepsilon}(E)\|$$

if $a > \frac{n}{p}$.

Hereby we obtain for $g := \left((1-\phi)|\cdot|^l e^{-|\cdot|}\hat{f}\right)\check{}$

$$\begin{aligned}\left\|\sup_{|x-y|\le d}\|((1-\phi)|\cdot|^l e^{-|\cdot|}\hat{f})\check{}(y)|E\|\ \big|L_p\right\| &\le \|((1-\phi)|\cdot|^l e^{-|\cdot|}\hat{f})\check{}(y)|B_{p,1}^{1+\frac{n}{p}}(E)\|\\ &\le c\|f|B_{p,q}^s(E)\|\end{aligned} \tag{73}$$

resp. $\le c\|f|F_{p,q}^s(E)\|$ because - as noted in the remarks prior to proposition 2.11 - the functions $(1-\phi)|\cdot|^l e^{-|\cdot|}(1+|\cdot|^2)^{\frac{\lambda}{2}}$ are Fourier multipliers for all spaces $B_{p,q}^s(E)$

and $F^s_{p,q}(E)$ and for all $\lambda \in \mathbb{R}$. If we put both results (72) and (73) into (71), we arrive at the desired estimate

$$\begin{aligned}\left\| \sup_{|x-y|\le d} \| \frac{\partial^l u(y,1)}{\partial t^l} \big| E \right\| &\le c\|(\phi \hat{f})\check{}\,|L_p(E)\| + \|f|B^s_{p,q}(E)\| \\ &\le c'\|f|B^s_{p,q}(E)\|\end{aligned}$$

and at an analogue for $F^s_{p,q}(E)$. □

3 Atomic characterizations of vector-valued function spaces

3.1 Atomic and harmonic representations

After dealing with the necessary arrangements we now take a look at atomic representations. It is our aim to represent every element of a function space $B^s_{p,q}(E)$ resp. $F^s_{p,q}(E)$ as a preferably easy (infinite) linear combination of "good-natured" functions. To this we describe the concept of atoms as one can find it in [Tri97], definition 13.3, p. 73. Thereby $Q_{\nu,m} := \{x \in \mathbb{R}^n : |x_i - 2^{-\nu}m_i| \leq 2^{-\nu-1}\}$ stands for the cube with sides parallel to the axes and with the center at $2^{-\nu}m$ and side length $2^{-\nu}$ for $m \in \mathbb{Z}^n$ and $\nu \in \mathbb{N}_0$.

Definition 3.1. (i) Let $K \in \mathbb{N}_0$ and $d > 1$. A K times differentiable (in the case $K = 0$ continuous) function $a : \mathbb{R}^n \to E$ is called (E-valued) 1-atom (more exactly 1_K-atom) if

$$\begin{aligned} \textit{supp}\ a \subset d \cdot Q_{0,m} &\text{ for an } m \in \mathbb{Z}, \\ \|D^\alpha a(x)|E\| \leq 1 &\text{ for all } |\alpha| \leq K. \end{aligned}$$

(ii) Let $s \in \mathbb{R}$, $0 < p \leq \infty$, $K \in \mathbb{N}_0$, $L+1 \in \mathbb{N}_0$ and $d > 1$. A K times differentiable (in the case $K = 0$ continuous) function $a : \mathbb{R}^n \to E$ is called (E-valued) (s,p)-atom (more exactly $(s,p)_{K,L}$-atom) if there exists a $\nu \in \mathbb{N}_0$ such that

$$\textit{supp}\ a \subset d \cdot Q_{\nu,m} \text{ for an } m \in \mathbb{Z}, \tag{74}$$

$$\|D^\alpha a(x)|E\| \leq 2^{-\nu\left(s-\frac{n}{p}\right)+|\alpha|\nu} \text{ for all } |\alpha| \leq K, \tag{75}$$

$$\int_{\mathbb{R}^n} x^\beta a(x)\, dx = 0 \text{ for all } |\beta| \leq L. \tag{76}$$

In particular, $a_{\nu,m}e_{\nu,m}$ is a vector-valued $(s,p)_{K,L}$-atom if $a_{\nu,m}$ is a scalar (i.e. $\mathbb{C}$-valued) $(s,p)_{K,L}$-atom and $e_{\nu,m} \in U_E = \{x \in E : \|x|E\| = 1\}$.

Furthermore, we introduce the sequence spaces $b_{p,q}$ and $f_{p,q}$ whose use will become clear in the following. At this we refer to [Tri97], definition 13.5, p. 74.

Definition 3.2. Let $0 < p \leq \infty$, $0 < q \leq \infty$ and

$$\lambda = \{\lambda_{\nu,m} \in \mathbb{C} : \nu \in \mathbb{N}_0, m \in \mathbb{Z}^n\}.$$

In addition, let

$$b_{p,q} := \left\{ \lambda : \|\lambda|b_{p,q}\| = \left(\sum_{\nu=0}^{\infty} \left(\sum_{m \in \mathbb{Z}^n} |\lambda_{\nu,m}|^p \right)^{\frac{q}{p}} \right)^{\frac{1}{q}} < \infty \right\}$$

and

$$f_{p,q} := \left\{ \lambda : \|\lambda|f_{p,q}\| = \left\| \left(\sum_{\nu=0}^{\infty} \sum_{m \in \mathbb{Z}^n} |\lambda_{\nu,m}\chi^{(p)}_{\nu,m}(\cdot)|^q \right)^{\frac{1}{q}} \Big| L_p \right\| < \infty \right\}$$

(modified in the case $p = \infty$ or $q = \infty$), where $\chi^{(p)}_{\nu,m}$ is the L_p-normalized characteristic function of the cube $Q_{\nu,m}$, i.e

$$\chi^{(p)}_{\nu,m} = 2^{\frac{\nu n}{p}} \text{ if } x \in Q_{\nu,m} \text{ and } \chi^{(p)}_{\nu,m} = 0 \text{ if } x \notin Q_{\nu,m}.$$

By observing

$$\|\lambda|b_{p,q}\| = \left(\sum_{\nu=0}^{\infty} \left\| \sum_{m\in\mathbb{Z}^n} |\lambda_{\nu,m}\chi^{(p)}_{\nu,m}(\cdot)|L_p \right\|^q \right)^{\frac{1}{q}}$$

it follows $b_{p,\min(p,q)} \hookrightarrow f_{p,q} \hookrightarrow b_{p,\max(p,q)}$ and $b_{p,p} = f_{p,p}$ in correspondence to the behaviour of the spaces $B^s_{p,q}(E)$ and $F^s_{p,q}(E)$.

The following lemma is oriented towards [Tri97], corollary 13.9, p. 81 which considers the scalar case. But we modify the original proof a bit. Here we get a first clue how the sought representation of all functions from $B^s_{p,q}(E)$ resp. $F^s_{p,q}(E)$ looks like.

Lemma 3.3. *Let $0 < p \le \infty$ resp. $< \infty$, $0 < q \le \infty$ and $s \in \mathbb{R}$. Let $K \in \mathbb{N}_0$, $L + 1 \in \mathbb{N}_0$ with*

$$K \ge 1 + \lfloor s \rfloor \text{ and } L \ge \lfloor \sigma_p - s \rfloor. \tag{77}$$

Then

$$\sum_{\nu=0}^{\infty} \sum_{m\in\mathbb{Z}^n} \lambda_{\nu,m} a_{\nu,m}(x)$$

converges unconditionally in $\mathcal{S}'(\mathbb{R}^n, E)$, where $a_{\nu,m}$ are E-valued 1_K-atoms (for $\nu = 0$) or E-valued $(s,p)_{K,L}$-atoms (for $\nu \in \mathbb{N}$) and $\lambda \in b_{p,q}$ or $\lambda \in f_{p,q}$.

Proof. Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$. In view of (76) we obtain

$$\int_{\mathbb{R}^n} \lambda_{\nu,m} a_{\nu,m}(x)\varphi(x)\, dx = \int_{\mathbb{R}^n} \lambda_{\nu,m} a_{\nu,m}(x) \left(\varphi(x) - \sum_{|\beta|\le L} c^{\nu,m}_{\beta} (x - 2^{-\nu}m)^{\beta} \right),$$

where $c^{\nu,m}_{\beta} \in \mathbb{C}$ is the coefficient for β in the Taylor expansion of φ at $2^{-\nu}m$. The modulus of the difference under the integral can be estimated from above (with arbitrary $M > 0$) by

$$\begin{aligned} &c\, 2^{-\nu(L+1)}(1 + |x|^2)^{-\frac{M}{2}} \sup_{y\in\mathbb{R}^n} (1 + |y|^2)^{\frac{M}{2}} \sum_{|\gamma|\le L+1} |D^{\gamma}\varphi(y)| \\ &\qquad = c\, 2^{-\nu(L+1)}(1 + |x|^2)^{-\frac{M}{2}} \|\varphi\|_{M,L+1} \end{aligned}$$

because it follows by (74) that

$$|\left(x - 2^{-\nu}m\right)^{\gamma}| \le c\, 2^{-\nu(L+1)} \text{ for } |\gamma| = L + 1 \text{ and } x \in \textit{supp}\ a_{\nu,m}.$$

In the case $1 \le p \le \infty$ by (77) we have $L+1 > -s$ and so by using (75)

$$2^{-\nu(L+1)}\|a_{\nu,m}|E\| \le 2^{-\nu\left(s-\frac{n}{p}\right)}2^{-\nu(L+1)} \le \chi^{(p)}_{\nu,m}2^{-\nu\varkappa}$$

with a $\varkappa > 0$. Keeping in mind that for fixed ν the supports of $a_{\nu,m}$ are "nearly" disjoint we obtain together with Hölder's inequality

$$\begin{aligned}
&\sum_m \left\| \int_{\mathbb{R}^n} \lambda_{\nu,m}a_{\nu,m}(x)\varphi(x)\ dx\Big|E \right\| \\
&\quad \le c\|\varphi\|_{M,L+1}\sum_m \int_{\mathbb{R}^n} 2^{-\nu(L+1)}\|\lambda_{\nu,m}a_{\nu,m}(x)|E\|(1+|x|^2)^{-\frac{M}{2}}\ dx \\
&\quad = c\|\varphi\|_{M,L+1}\int_{\mathbb{R}^n}\sum_m 2^{-\nu(L+1)}\|\lambda_{\nu,m}a_{\nu,m}(x)|E\|(1+|x|^2)^{-\frac{M}{2}}\ dx \\
&\quad \le c'\|\varphi\|_{M,L+1}\int_{\mathbb{R}^n}\left(\sum_m \left(2^{-\nu(L+1)}\|\lambda_{\nu,m}a_{\nu,m}(x)|E\|\right)^p\right)^{\frac{1}{p}}(1+|x|^2)^{-\frac{M}{2}}\ dx \\
&\quad \le c''\|\varphi\|_{M,L+1}\left(\int_{\mathbb{R}^n}\sum_m \left(2^{-\nu(L+1)}\|\lambda_{\nu,m}a_{\nu,m}(x)|E\|\right)^p\right)^{\frac{1}{p}} \\
&\quad = c''\|\varphi\|_{M,L+1}\left(\sum_m \int_{d\cdot Q_{\nu,m}} \left(2^{-\nu(L+1)}\|\lambda_{\nu,m}a_{\nu,m}(x)|E\|\right)^p\ dx\right)^{\frac{1}{p}} \\
&\quad \le c''2^{-\nu\varkappa}\|\varphi\|_{M,L+1}\left(\sum_m |\lambda_{\nu,m}|^p\int_{d\cdot Q_{\nu,m}} 2^{\nu n}\ dx\right)^{\frac{1}{p}} \\
&\quad \le c'''2^{-\nu\varkappa}\|\varphi\|_{M,L+1}\left(\sum_m |\lambda_{\nu,m}|^p\right)^{\frac{1}{p}}.
\end{aligned}$$

if we only choose M so large that $Mp' > n$. Because of $\varkappa > 0$ it follows

$$\begin{aligned}
\sum_\nu\sum_m \left\| \int_{\mathbb{R}^n} \lambda_{\nu,m}a_{\nu,m}(x)\varphi(x)\ dx\Big|E \right\| &\le c'''\ \|\varphi\|_{M,L+1}\sum_\nu 2^{-\nu\varkappa}\left(\sum_m |\lambda_{\nu,m}|^p\right)^{\frac{1}{p}} \\
&\le C\ \|\varphi\|_{M,L+1}\sup_\nu\left(\sum_m |\lambda_{\nu,m}|^p\right)^{\frac{1}{p}} \\
&= C\ \|\varphi\|_{M,L+1}\cdot\|\lambda_{\nu,m}|b_{p,\infty}\|.
\end{aligned} \tag{78}$$

Because of $b_{p,q} \hookrightarrow b_{p,\infty}$ we have shown the absolute convergence of the above series in the Banach space E. Hence the series itself converges unconditionally in the Banach space E. This shows the desired claim by an admissible commutation of the integral and the sums.

In the case $0 < p < 1$ we have $L+1 > -s + \frac{n}{p} - n$ by the assumptions instead and hence

$$2^{-\nu(L+1)}\|a_{\nu,m}|E\| \le c\,2^{-\nu\left(s-\frac{n}{p}\right)}2^{-\nu(L+1)} \le c\,2^{\nu n}2^{-\nu\varkappa}.$$

So we obtain the above estimate (78) for $p = 1$ and because of $b_{p,q} \hookrightarrow b_{1,q} \hookrightarrow b_{1,\infty}$ the convergence in $\mathcal{S}'(\mathbb{R}^n, E)$ for $p < 1$ follows as well.

If $\lambda \in f_{p,q}$, then $\lambda \in b_{p,\infty}$ and hence the convergence in $\mathcal{S}'(\mathbb{R}^n, E)$ follows from the facts proven above.

Furthermore note that the condition $K = 0$ would have sufficed for the whole proof, i.e. taking continuous atoms with suitable boundary conditions without any restrictions on the derivatives but with possible moment conditions. □

In the following we prove a characterization of such sums as elements of the function spaces $B^s_{p,q}(E)$ and $F^s_{p.q}(E)$. At this we stick to [Tri97], theorem 13.8, p. 75, step 2, which treats the scalar case. In the last part of the proof we give a slight modification due to a small gap regarding the maximal function in the original proof. Otherwise the proof can be taken over nearly verbatim.

Proposition 3.4. *(i) Let $0 < p \le \infty$, $0 < q \le \infty$ and $s \in \mathbb{R}$. Let $K \in \mathbb{N}_0$ and $L + 1 \in \mathbb{N}_0$ with*

$$K \ge 1 + \lfloor s \rfloor \text{ and } L \ge \lfloor \sigma_p - s \rfloor.$$

Then every $f \in \mathcal{S}'(\mathbb{R}^n, E)$ which can be represented by

$$f = \sum_{\nu \in \mathbb{N}_0} \sum_{m \in \mathbb{Z}^n} \lambda_{\nu,m} a_{\nu,m}$$

in $\mathcal{S}'(\mathbb{R}^n, E)$ belongs to $B^s_{p,q}(E)$. Thereby $a_{\nu,m}$ are E-valued 1_K-atoms (for $\nu = 0$) or E-valued $(s,p)_{K,L}$-atoms (for $\nu \in \mathbb{N}$) and $\lambda \in b_{p,q}$. Furthermore, there exists a constant c independent of f, λ and $a_{\nu,m}$, i.e. independent of the found representation of f such that

$$\|f|B^s_{p,q}(E)\| \le c\|\lambda|b_{p,q}\|.$$

(ii) Let $0 < p < \infty$, $0 < q \le \infty$ and $s \in \mathbb{R}$. Let $K \in \mathbb{N}_0$ and $L + 1 \in \mathbb{N}_0$ with

$$K \ge 1 + \lfloor s \rfloor \text{ and } L \ge \lfloor \sigma_{p,q} - s \rfloor. \tag{79}$$

Then every $f \in \mathcal{S}'(\mathbb{R}^n, E)$ which can be represented by

$$f = \sum_{\nu \in \mathbb{N}_0} \sum_{m \in \mathbb{Z}^n} \lambda_{\nu,m} a_{\nu,m}$$

in $\mathcal{S}'(\mathbb{R}^n, E)$ belongs to $F^s_{p,q}(E)$. Thereby $a_{\nu,m}$ are E-valued 1_K-atoms (for $\nu = 0$) or E-valued $(s,p)_{K,L}$-atoms (for $\nu \in \mathbb{N}$) and $\lambda \in f_{p,q}$. Furthermore, there exists a constant c independent of f, λ and $a_{\nu,m}$, i.e. independent of the found representation of f such that

$$\|f|F^s_{p,q}(E)\| \le c\|\lambda|f_{p,q}\|.$$

Proof. In the proof we rely on the equivalent quasi-norm from proposition 2.9 which results from theorem 2.1. We choose the functions $k_0, k^0 \in \mathcal{S}(\mathbb{R}^n)$ and hence also $k^N := \Delta^N k^0$ so that they have compact support, i.e. *supp* k_0, *supp* $k^0 \subset e \cdot B$ for an

$e > 0$. Let $a_{\nu,m}$ with $\nu \in \mathbb{N}$ and $m \in \mathbb{Z}^n$ be an E-valued $(s,p)_{K,L}$-atom by definition 3.1. If $\nu = 0$, let $a_{\nu,m}$ be an E-valued 1_K-atom. Then for $j \in \mathbb{N}$ it holds

$$2^{js}\left(k_j^N * a_{\nu,m}\right)(x) = 2^{js}\int_{\mathbb{R}^n} \Delta^N k^0(y) a_{\nu,m}(x - 2^{-j}y))\, dy.$$

We distinguish in the following between the two cases $j \geq \nu$ and $j < \nu$. We will include the boundary cases in which we have $\nu = 0$ or k_0 instead of k^N at suitable points.

Let $j \geq \nu$. Then

$$a^{\nu,m}(x) := 2^{\nu\left(s-\frac{n}{p}\right)} a_{\nu,m}(2^{-\nu}(x+m))$$

is a 1_K-atom with *supp* $a^{\nu,m} \subset d \cdot B$. If $K = 2M$, $M \in \mathbb{N}_0$, we obtain by partial integration

$$\begin{aligned} 2^{js}\left(k_j^N * a_{\nu,m}\right)(x) &= 2^{js}2^{-\nu\left(s-\frac{n}{p}\right)}\int_{\mathbb{R}^n} \Delta^N k^0(y) a^{\nu,m}(2^\nu x - m - 2^{\nu-j}y)\, dy \\ &= 2^{(s-K)(j-\nu)}2^{\nu\frac{n}{p}}\int_{\mathbb{R}^n} \Delta^{N-M}k^0(y)\cdot\Delta^M a^{\nu,m}(2^\nu x - m - 2^{\nu-j}y)\, dy. \end{aligned}$$

If K is odd, one has to modify this in an obvious way. If we take a look at the supports of the functions k^0 and $a^{\nu,m}$ (see (74)), we see that the integral vanishes if $|2^\nu x - m + 2^{\nu-j}y| > d$ for all $y \in \mathbb{R}^n$ with $|y| \leq e$. For all other x the integral is bounded by a $c > 0$ independent of j and ν, which can be seen by (75) and the properties of k_0. Therefore, we obtain

$$\begin{aligned} 2^{js}\|\left(k_j^N * a_{\nu,m}\right)(x)|E\| &\leq c\, 2^{-(K-s)(j-\nu)}\tilde{\chi}^{(p)}_{\nu,m}(x) \\ &= c\, 2^{-\varkappa(j-\nu)}\tilde{\chi}^{(p)}_{\nu,m}(x), \end{aligned} \tag{80}$$

where $\tilde{\chi}^{(p)}_{\nu,m}(x)$ is the L_p-normalised characteristic function of the cube $c \cdot Q_{\nu,m}$ and $\varkappa > 0$ by (79). The case $\nu = 0$, i.e. the case in which $a_{\nu,m}$ is a 1_K-atom can be treated in the same way.

Now let $j < \nu$. It follows by a transformation of variables

$$2^{js}\left(k_j^N * a_{\nu,m}\right)(x) = 2^{j(s+n)}\int_{|y|\leq e2^{-j}} k^N(2^j y) a_{\nu,m}(x-y)\, dy. \tag{81}$$

We expand the function $k^N(2^j\cdot)$ into a Taylor series in the point $2^{-\nu}m + x$ to the order L, (in case of $F^s_{p,q}(E)$) given by (79), and obtain

$$k^N(2^j y) = \sum_{|\beta|\leq L} c_\beta(x)(y - 2^{-\nu}m - x)^\beta + 2^{j(L+1)}R_x(y)$$

with

$$\frac{|R_x(y)|}{|y - 2^{-\nu}m - x|^{L+1}} \leq c \sum_{|\alpha|=L+1}\sup_{y\in\mathbb{R}^n}|D^\alpha k^N(y)| \leq c'.$$

If we put this into (81), the terms with $|\beta| \leq L$ vanish because $a_{\nu,m}$ is a $(s,p)_{K,L}$-atom (see (76)). Furthermore, by (74) and (75) we have

$$\|a_{\nu,m}(x-y)|E\| \leq 2^{-\nu\left(s-\frac{n}{p}\right)}\tilde{\chi}_{\nu,m}(x-y),$$

where $\tilde{\chi}_{\nu,m}$ is the characteristic function of $d\cdot Q_{\nu,m}$ and $|y-2^{-\nu}m-x|^{L+1} \leq c2^{-\nu(L+1)}$ for $x-y \in supp\ a_{\nu,m}$. From that it follows

$$2^{js}\| \left(k_j^N * a_{\nu,m}\right)(x)|E\| \leq c2^{j(s+n)}2^{-\nu\left(s-\frac{n}{p}\right)}2^{(j-\nu)(L+1)} \int_{|y|\leq e2^{-j}} \tilde{\chi}_{\nu,m}(x-y)\ dy. \quad (82)$$

In the case $j=0$ - this means if we convolute with k_0 instead of k^N - we can use the same calculation.

Now the integral on the right-hand side is at most $d^n 2^{-\nu n}$ and vanishes if we have $|x-2^{-\nu}m| > d2^{-\nu}+e2^{-j}$, i. e. if $x \notin c2^{\nu-j}\cdot Q_{\nu,m}$ for a suitable $c>0$, observing $j<\nu$. Altogether the integral is smaller or equal to $d^n 2^{-\nu n}\chi(c2^{\nu-j}Q_{\nu,m})(x)$. Now we work with the Hardy-Littlewood maximal function (see (4)). We have

$$\begin{aligned} \sum_m |\lambda_{\nu,m}| \int_{|y|\leq e2^{-j}} \tilde{\chi}_{\nu,m}(x-y)\ dy &\leq \sum_m |\lambda_{\nu,m}| d^n 2^{-\nu n}\chi(c2^{\nu-j}Q_{\nu,m})(x) \\ &= d^n 2^{-\nu n} \sum_{m\in D_x} |\lambda_{\nu,m}|, \end{aligned} \quad (83)$$

where $D_x := \{m \in \mathbb{Z}^n : x \in c\,2^{\nu-j}Q_{\nu,m}\}$. Then let

$$E_x := \bigcup_{m\in D_x} c2^{\nu-j}Q_{\nu,m}.$$

There is a constant $c' > c$ independent of m and ν such that $E_x \subset B_{c'2^{-j}}(x)$. Simultaneously it holds

$$\begin{aligned} M\left(\sum_m |\lambda_{\nu,m}|\chi(Q_{\nu,m})\right)^w &\geq \frac{1}{|B_{c'2^{-j}}(x)|}\int_{B_{c'2^{-j}}(x)} \left(\sum_m |\lambda_{\nu,m}|\chi(Q_{\nu,m})(y)\right)^w dy \\ &\geq c''2^{jn}\int_{B_{c'2^{-j}}(x)} \left(\sum_{m\in D_x} |\lambda_{\nu,m}|\chi(Q_{\nu,m})(y)\right)^w dy \\ &\geq c''2^{(j-\nu)n}\sum_{m\in D_x} |\lambda_{\nu,m}|^w \end{aligned}$$

because $Q_{\nu,m} \subset E_x$ for $m \in D_x$ and the $Q_{\nu,m}$ are pairwise disjoint. Together with (83) (observing $w<1$) this yields

$$\sum_m |\lambda_{\nu,m}| \int_{|y|\leq e2^{-j}} \tilde{\chi}_{\nu,m}(x-y)\ dy \leq C2^{-\nu n}2^{(\nu-j)\frac{n}{w}}\left(M\left(\sum_m |\lambda_{\nu,m}|\chi(Q_{\nu,m})\right)^w\right)^{\frac{1}{w}}(x).$$

If we put this into (82) and replace the characteristic function by the L_p-normalized characteristic function, we can conclude

$$\begin{aligned} &2^{js}\left\|\left(k_j^N*\sum_m|\lambda_{\nu,m}|a_{\nu,m}\right)(x)\big|E\right\| \\ &\le c\,2^{j(s+n)}2^{-\nu\left(s-\frac{n}{p}\right)}2^{(j-\nu)(L+1)}2^{-\nu n}2^{(\nu-j)\frac{n}{w}}\cdot\left(M\left(\sum_m|\lambda_{\nu,m}|\chi(Q_{\nu,m})\right)^w\right)^{\frac{1}{w}}(x) \\ &=c\,2^{-(\nu-j)(L+1+s+n-\frac{n}{w})}\left(M\left(\sum_m|\lambda_{\nu,m}|\chi^{(p)}_{\nu,m}\right)^w\right)^{\frac{1}{w}}(x) \\ &=c\,2^{-(\nu-j)\varkappa}\left(M\left(\sum_m|\lambda_{\nu,m}|\chi^{(p)}_{\nu,m}\right)^w\right)^{\frac{1}{w}}(x). \end{aligned} \tag{84}$$

Here we have $\varkappa>0$ if we choose w close enough to $\min(1,p,q)$ resp. $\min(1,p)$. This is possible because of (79) in the case $F^s_{p,q}(E)$ resp. the analogous condition for $B^s_{p,q}(E)$ and the definition of σ_p resp. $\sigma_{p,q}$ (see (21)). If we now use (80) and (84), we obtain (keeping lemma 3.3 and $k_j^N\in\mathcal{S}(\mathbb{R}^n)$ in mind)

$$\begin{aligned} 2^{js}\left\|\left(k_j^N*\sum_{\nu,m}\lambda_{\nu,m}a_{\nu,m}(x)\right)\big|E\right\| &\le c\sum_{\nu\le j,m}2^{-|j-\nu|\varkappa}|\lambda_{\nu,m}|\tilde{\chi}^{(p)}_{\nu,m}(x) \\ &\quad+c\sum_{\nu>j}2^{-|j-\nu|\varkappa}\left(M\left(\sum_m|\lambda_{\nu,m}|\chi^{(p)}_{\nu,m}\right)^w\right)^{\frac{1}{w}}(x). \end{aligned}$$

Now we can apply lemma 2.4. We put ($k_0^N:=k_0$)

$$G_j:=2^{js}\left\|\left(k_j^N*\sum_{\nu,m}\lambda_{\nu,m}a_{\nu,m}(x)\right)\big|E\right\|$$

and

$$g_\nu:=\sum_m|\lambda_{\nu,m}|\tilde{\chi}^{(p)}_{\nu,m}(x)+\left(M\left(\sum_m|\lambda_{\nu,m}|\chi^{(p)}_{\nu,m}\right)^w\right)^{\frac{1}{w}}(x).$$

Then it follows (with triangle inequality and the "almost"-disjointness of the $c\cdot Q_{\nu,m}$)

$$\begin{aligned} &\left\|\left(\sum_{j=0}^\infty 2^{jsq}\left\|\left(k_j^N*\sum_{\nu,m}\lambda_{\nu,m}a_{\nu,m}\right)\big|E\right\|^q\right)^{\frac{1}{q}}\big|L_p\right\| \\ &\le C\left\|\left(\sum_{\nu,m}\left(|\lambda_{\nu,m}|\ \tilde{\chi}^{(p)}_{\nu,m}\right)^q\right)^{\frac{1}{q}}\big|L_p\right\|+C\left\|\left(\sum_\nu\left(M\left(\sum_m|\lambda_{\nu,m}|\chi^{(p)}_{\nu,m}\right)^w\right)^{\frac{q}{w}}\right)^{\frac{1}{q}}\big|L_p\right\| \end{aligned}$$

and an analogous result for $B^s_{p,q}(E)$. Furthermore, if we keep

$$\tilde{\chi}^{(p)}_{\nu,m} \leq C' \left(M\big(\chi^{(p)}_{\nu,m}\big)^w \right)^{\frac{1}{w}} \tag{85}$$

in mind for a C' independent of ν and m, then the first and the second term are estimated by nearly the same terms. The formula (85) can be derived analogously (but simpler, $\nu = j$) to (83) and the following step.

If we now use $\| \, \|M(f_n^w)^{\frac{1}{w}}|l_q\| \; |L_p\| = \| \, \|M(f_n^w)|l_{\frac{q}{w}}\| \; |L_{\frac{p}{w}}\|^{\frac{1}{w}}$ and the fact that $w < p$ and $w < q$, we obtain our desired result by the boundedness of the maximal operator from $L_r(l_s)$ to $L_r(l_s)$ for $s, r > 1$ and $r < \infty$ (see (7)). In the case $B^s_{p,q}(E)$ we only need the boundedness of the maximal operator from $l_s(L_r)$ to $l_s(L_r)$, which is given for $r > 1$ (see (6)) such that $w < p$ and hence $L \geq \lfloor \sigma_p - s \rfloor$ suffices. □

We have shown that every $f \in \mathcal{S}'(\mathbb{R}^n, E)$ which can be represented in the given way is an element of the function space $B^s_{p,q}(E)$ resp. $F^s_{p,q}(E)$. Now the question will be whether all elements of the function space can be represented in such a way. The positive answer in the scalar (i.e. $E = \mathbb{C}$) case has been given for instance in [Tri97], theorem 13.8, p. 75. For the vector-valued case we will slightly alter the derivation sequence, as described in [Tri97], theorem 15.8, p. 114 . In the first step we care about a representation with harmonic, vector-valued atoms. This is inspired by the norms from proposition 2.11, in which the functions $u(x, t)$ are harmonic in the domain $\{x \in \mathbb{R}^n, t > 0\}$.

To explain this a bit more in detail (as in [Tri97], section 12.2, p. 59 in the scalar case) we choose an $f \in \mathcal{S}(\mathbb{R}^n, E)$ and form the functions $u(x, t)$ for $x \in \mathbb{R}^n$, $t > 0$ as in example 2.10 by

$$u(x,t) := (e^{-t|\cdot|}\hat{f})^{\check{}}(x) = d_n \left(f * \frac{t}{(|\cdot|^2 + t^2)^{\frac{n+1}{2}}} \right)(x).$$

We obtain

$$u(x,t) \to f(x) \text{ for } t \to 0$$

uniformly in x because $(e^{-|\cdot|})^{\check{}} \in L_1$ and $e^{-|0|} = 1$. As in the scalar case this follows from

$$\begin{aligned} u(x,t) - f(x) &= \int_{y \in \mathbb{R}^n} (f(x-y) - f(x)) \frac{t^{-n}}{(|\frac{y}{t}|^2 + 1)^{\frac{n+1}{2}}} \, dy \\ &= \int_{y \in \mathbb{R}^n} (f(x-ty) - f(x)) \frac{1}{(|y|^2 + 1)^{\frac{n+1}{2}}} \, dy \\ &= \int_{|ty| \leq \delta} (f(x-ty) - f(x)) \frac{1}{(|y|^2 + 1)^{\frac{n+1}{2}}} \, dy \\ &\quad + \int_{|ty| > \delta} (f(x-ty) - f(x)) \frac{1}{(|y|^2 + 1)^{\frac{n+1}{2}}} \, dy. \end{aligned}$$

Furthermore, we have

$$t^k \frac{\partial^k u(x,t)}{\partial t^k} \to 0 \text{ for } t \to 0$$

uniformly in x for all $k \in \mathbb{N}$ because $(|\cdot|^k e^{-|\cdot|})^\check{} \in L_1$ and $|0|^k e^{-|0|} = 0$. Now we obtain by iterated partial integration and with suitable constants d_l^k with $k \in \mathbb{N}$ and $l \in \{0, \dots, k-1\}$

$$\begin{aligned} \int_a^b t^{k-1} \frac{\partial^k u(x,t)}{\partial t^k}\, dt &= \tau^{k-1} \frac{\partial^{k-1} u(x,\tau)}{\partial \tau^{k-1}}\Big|_a^b - (k-1) \int_a^b t^{k-2} \frac{\partial^{k-1} u(x,t)}{\partial t^{k-1}} \\ &= \dots \\ &= \sum_{l=0}^{k-1} d_l^k b^l \frac{\partial^l u(x,b)}{\partial t^l} - \sum_{l=0}^{k-1} d_l^k a^l \frac{\partial^l u(x,a)}{\partial t^l}. \end{aligned} \tag{86}$$

Therefore, we get by

$$\int_0^1 t^{k-1} \frac{\partial^k u(x,t)}{\partial t^k}\, dt = \sum_{\nu=0}^{\infty} \int_{2^{-\nu-1}}^{2^{-\nu}} t^{k-1} \frac{\partial^k u(x,t)}{\partial t^k}\, dt$$

the relation

$$\sum_{l=0}^{k-1} d_l^k \tau^l \frac{\partial^l u(x,\tau)}{\partial t^l}\Big|_0^1 = \sum_{\nu=0}^{\infty} \int_{2^{-\nu-1}}^{2^{-\nu}} t^{k-1} \frac{\partial^k u(x,t)}{\partial t^k}\, dt.$$

By our above considerations on the limits it follows

$$f(x) = c \sum_{\nu=0}^{\infty} \int_{2^{-\nu-1}}^{2^{-\nu}} t^{k-1} \frac{\partial^k u(x,t)}{\partial t^k}\, dt + \sum_{l=0}^{k-1} c_l^k \frac{\partial^l u(x,1)}{\partial t^l} \tag{87}$$

with suitable constants c_l^k with $k \in \mathbb{N}$ and $l \in \{0, \dots, k-1\}$. In particular, we have $c_0^k = 1$. We want to call the right-hand side a harmonic representation of f. A look at the norms from proposition 2.11 tells us that very similar terms occured there. Therefore, it will be our aim to give (87) a meaning for $f \in B_{p,q}^s(E)$ resp. $F_{p,q}^s(E)$, with convergence at least in $\mathcal{S}'(\mathbb{R}^n, E)$. From the remarks forward to 2.11 we obtain that the functions $u(x,t)$ are well-defined for $f \in B_{p,q}^s(E)$ resp. $F_{p,q}^s(E)$, harmonic in $\{(x,t) \in \mathbb{R}^{n+1}, t > 0\}$ and bounded on $\{(x,t) \in \mathbb{R}^{n+1}, t > \delta\}$ for every $\delta > 0$. So the integrals in (87) make sense.

In the following we keep close to [Tri97], theorem 12.5, p. 62, where the scalar case is treated.

Proposition 3.5. *Let $s \in \mathbb{R}$, $0 < q \le \infty$ and $0 < p \le \infty$ (resp. $< \infty$). If one chooses $k \in \mathbb{N}$ large enough, then the right-hand side of* (87) *converges in $\mathcal{S}'(\mathbb{R}^n, E)$ to f for $f \in B_{p,q}^s(E)$ resp. $F_{p,q}^s(E)$.*

Proof. Let

$$\varphi_0(\xi) := \left(\sum_{l=0}^{k-1} (-1)^l c_l^k |\xi|^l \right) e^{-|\xi|}$$

with the constants c_l^k from (87). Then it is

$$(\varphi_0(t\cdot)\hat{f})^\check{} = \sum_{l=0}^{k-1} c_l^k t^k \frac{\partial^l u(x,t)}{\partial t^l}$$

for $f \in B^s_{p,q}(E)$ resp. $f \in F^s_{p,q}(E)$ by the definitions and remarks in example 2.10. We consider the relationship from (86) for $b = 1$ and $a = t$. To ensure that the right-hand side of (87) converges in $\mathcal{S}'(\mathbb{R}^n, E)$ at all for $t \to 0$, it suffices to show that $(\varphi_0(t\cdot)\hat{f})\check{}$ converges in $\mathcal{S}'(\mathbb{R}^n, E)$. But this is the case if the left-hand side of (86) converges in $\mathcal{S}'(\mathbb{R}^n, E)$.

If $\psi \in \mathcal{S}(\mathbb{R}^n)$ and $f \in B^s_{p,q}(E)$ or $F^s_{p,q}(E)$, it follows

$$\left[t|\cdot|^k e^{-t|\cdot|}\hat{f}\right]^\vee(\psi) = \left(t|\cdot|^k e^{-t|\cdot|}\left(\psi(-\cdot) * f\right)\hat{}\right)^\vee(0),$$

so

$$\left[\int_a^1 t^{k-1}\frac{\partial^k u(x,t)}{\partial t^k}\,dt\right](\psi) = \int_a^1 t^{k-1}\frac{\partial^k \tilde{u}(0,t)}{\partial t^k}\,dt,$$

where the function $\tilde{u}$ is the function u build with $\psi(-\cdot) * f$ instead of f. Hence it suffices to show that the left-hand side converges for $t \to 0$ for functions of the form $\psi * f$ with $\psi \in \mathcal{S}(\mathbb{R}^n)$ and $f \in B^s_{p,q}(E)$ resp. $F^s_{p,q}(E)$ in $L_\infty(E)$. But this can be obtained by proposition 2.11 if we choose k large enough. Then it holds (in the case $B^s_{p,q}(E)$)

$$\int_0^1 t^k\|\frac{\partial^k \tilde{u}(\cdot,t)}{\partial t^k}|L_\infty(E)\|\,\frac{dt}{t} \le c\|\psi * f|B^0_{\infty,1}(E)\| \le c'\|f|B^s_{p,q}(E)\|$$

for all $0 < p \le \infty$, $0 < q \le \infty$ and $s \in \mathbb{R}$ because $(1+|\cdot|^2)^{\frac{\sigma}{2}}\hat{\psi}$ are Fourier multipliers for $B^s_{p,q}(E)$ for all $\sigma \in \mathbb{R}$ (see (16)) and because of the Sobolev embeddings (see (18)). [7]

So it follows that there is a $g \in \mathcal{S}'(\mathbb{R}^n, E)$ such that $(\varphi_0(t\cdot)\hat{f})\check{} \to g$ for $t \to 0$. Now it remains to prove that $g = f$. Then we would have proven the representation (87) at least in $\mathcal{S}'(\mathbb{R}^n, E)$ for $f \in B^s_{p,q}(E)$ resp. $F^s_{p,q}(E)$. At first, let $f \in L_1(E)$ be assumed. As in our preliminary thoughts it follows

$$u(x,t) \to f \text{ for } t \to 0,$$
$$t^k\frac{\partial^k u(x,t)}{\partial t^k} \to 0 \text{ for } t \to 0$$

in $L_1(E)$ for all $k \in \mathbb{N}$. This yields $f = g$ under the given conditions.

If $a \in E'$ with $\|a|E'\| = 1$ and $g \in \mathcal{S}'(\mathbb{R}^n, E)$, then $a[g]$, defined for $\varphi \in \mathcal{S}(\mathbb{R}^n)$ by

$$a[g](\varphi) := a(g(\varphi)),$$

is an element of $\mathcal{S}'(\mathbb{R}^n)$. The linear mapping $g \mapsto a[g]$ from $\mathcal{S}'(\mathbb{R}^n, E)$ to $\mathcal{S}'(\mathbb{R}^n)$ is obviously continous. If $g \in B^s_{p,q}(E)$ resp. $F^s_{p,q}(E)$, then because of

$$|\varphi * a[g](x)| = |a[g](\varphi(x-\cdot))| \le \|g(\varphi(x-\cdot))|E\| = \|\left(\varphi * g\right)(x)|E\|$$

it is valid that $a[g] \in B^s_{p,q}$ resp. $F^s_{p,q}$. If $a[g] = a[f]$ for all $a \in E'$ with $\|a|E'\| = 1$, then we have $f = g$.

[7] Another possible proof can be given by the fact that the integrand is bounded in x and t since $\psi * f \in L_\infty$

We follow the third step of the proof in the scalar case in [Tri97], theorem 12.5, p. 62. Let $f \in B^s_{p,q}(E)$ resp. $F^s_{p,q}(E)$. By the Sobolev embeddings (see (18) and (19)) there exists a $\sigma \in \mathbb{R}$ such that $f \in \mathscr{C}^\sigma(E) = B^\sigma_{\infty,\infty}(E)$. Hence it is enough to prove the assertion for $f \in \mathscr{C}^\sigma(E)$ for all $\sigma \in \mathbb{R}$.

We use the (scalar) duality relation

$$\left(B^{-\sigma}_{1,1}\right)' = \mathscr{C}^\sigma, \tag{88}$$

see [Tri83], section 2.11.2, p. 178. If $\chi \in \mathcal{S}(\mathbb{R}^n)$ and if *supp* $\hat{\chi}$ is compact, then it follows for $E = \mathbb{C}$ from the fact we have just proven that

$$(\varphi_0(t\cdot)\hat{\chi})^\vee \to \chi \text{ in } L_1 \text{ for } t \to 0.$$

Because *supp* $\hat{\chi}$ is compact, we obtain

$$(\varphi_0(t\cdot)\check{\chi})^\wedge \to \chi \text{ in } B^{-\sigma}_{1,1} \text{ für } t \to 0. \tag{89}$$

Therefore, we obtain from $(\varphi_0(t\cdot)\hat{f})^\vee \to g$ in $\mathcal{S}'(\mathbb{R}^n, E)$, (88), (89) and $a[f] \in \mathscr{C}^\sigma$

$$\begin{aligned} a[g](\chi) &= \lim_{t\to 0} a\left((\varphi_0(t\cdot)\hat{f})^\vee(\chi)\right) \\ &= \lim_{t\to 0} a\left(f((\varphi_0(t\cdot)\check{\chi})^\wedge)\right) = \lim_{t\to 0} a[f]\left((\varphi_0(t\cdot)\check{\chi})^\wedge\right) = a[f](\chi). \end{aligned}$$

Hereby follows $a[f] = a[g]$ and finally$f = g$ since the set of functions $\chi \in \mathcal{S}(\mathbb{R}^n)$ whose Fourier transform has compact support is dense in $\mathcal{S}(\mathbb{R}^n)$. □

Now we derive a representation of the elements of the function spaces $B^s_{p,q}(E)$ resp. $F^s_{p,q}(E)$ which leads us to the desired form of proposition 3.4. Hereby we keep close to [Tri97], section 13.10, p. 83.

Let $f \in B^s_{p,q}(E)$ resp. $F^s_{p,q}(E)$. If we choose k sufficiently large, then we obtain the harmonic representation of proposition 3.5

$$f(x) = c\sum_{\nu=0}^{\infty}\int_{2^{-\nu-1}}^{2^{-\nu}} t^k \frac{\partial^k u(x,t)}{\partial t^k}\,\frac{dt}{t} + \sum_{l=0}^{k-1} c_l^k \frac{\partial^l u(x,1)}{\partial t^l}$$

with convergence in $\mathcal{S}'(\mathbb{R}^n, E)$.

Let $\mu \in \mathbb{N}$ be fixed. Let $\nu \in \mathbb{N}$, $\nu \geq \mu$, $m \in \mathbb{Z}^n$ and $l \in \{0, \cdots 2^\mu - 1\}$. By $B_{\nu,m,l}$ we denote the cubes in $\mathbb{R}^{n+1}_+ := \{(x,t) \in \mathbb{R}^{n+1}, t > 0\}$ with center $(2^{-\nu}m, 2^{-\nu+\mu} + l2^{-\nu})$ and radius $2^{-\nu+\mu-2}$ and by $Q_{\nu,m}$ the formerly introduced cubes. We decompose the rectangles $Q_{\nu,m} \times (2^{-\nu+\mu}, 2^{-\nu+\mu+1})$ in 2^μ cubes of side length $2^{-\nu}$ such that t is located betweeen $2^{-\nu+\mu} + l2^{-\nu}$ and $2^{-\nu+\mu} + (l+1)2^{-\nu}$ in the l-th cube. There is an obvious connection between $B_{\nu,m,l}$ and these cubes. Now we define

$$\lambda_{\nu,m} := 2^{\nu\left(s-\frac{n}{p}\right)} 2^{-\nu k} \sup \|\frac{\partial^k u(y,t)}{\partial t^k}|E\| \text{ for } \nu > \mu, m \in \mathbb{Z}^n, \tag{90}$$

where we take the supremum over the set

$$\left\{(y,t) \in \mathbb{R}^{n+1} : |2^{-\nu}m - y| \leq d2^{-\nu+\mu-1}, d^{-1}2^{-\nu+\mu} \leq t \leq d2^{-\nu+\mu+1}\right\},$$

for a $d > 0$ which we will choose sufficiently large afterwards.

In the case $\mu = \nu$ we put

$$\lambda_{\mu,m} := \sum_{l=0}^{k-1} |c_l^k| \sup \|\frac{\partial^l u(y,t)}{\partial t^l}|E\| \text{ for } m \in \mathbb{Z}^n,$$

where we take the supremum over the set

$$\left\{(y,t) \in \mathbb{R}^{n+1} : |2^{-\nu}m - y| \leq d, \frac{1}{2d} \leq t \leq \frac{3}{2}d\right\}.$$

Now we take a closer look at $\|\lambda|f_{p,q}\|$. We obtain

$$\begin{aligned}
\|\lambda|f_{p,q}\| &= \left\|\left(\sum_{\nu=\mu}^{\infty}\sum_{m\in\mathbb{Z}^n}|\lambda_{\nu,m}\chi_{\nu,m}^{(p)}|^q\right)^{\frac{1}{q}}\Big|L_p\right\| \\
&\sim \left\|\left(\sum_{\nu=\mu+1}^{\infty}\sum_{m\in\mathbb{Z}^n}\left(2^{\nu\left(s-\frac{n}{p}\right)}2^{-\nu k}\sup\|\frac{\partial^k u(y,t)}{\partial t^k}|E\|\ \chi_{\nu,m}^{(p)}\right)^q\right)^{\frac{1}{q}}\Big|L_p\right\| \\
&\quad + \left\|\sum_{m\in\mathbb{Z}^n}\sum_{l=0}^{k-1}|c_l^k|\sup\|\frac{\partial^l u(y,t)}{\partial t^l}|E\|\cdot\chi_{\mu,m}^{(p)}\ \Big|L_p\right\| \\
&\leq 2^{\mu(s-k)}\left\|\left(\sum_{\nu=\mu+1}^{\infty}\left(\int_{2^{-\nu+\mu}}^{2^{-\nu+\mu+1}}t^{(k-s)q}\ \widehat{\sup}\|\frac{\partial^k u(y,\tau)}{\partial t^k}|E\|^q\ \frac{dt}{t}\right)\right)^{\frac{1}{q}}\Big|L_p\right\| \\
&\quad + 2^{\mu\frac{n}{p}}\sum_{l=0}^{k-1}\left\|\sup_{|\cdot-y|\leq d',\frac{1}{2d}\leq t\leq\frac{3}{2}d}\|\frac{\partial^l u(y,t)}{\partial t^l}|E\|\ \Big|L_p\right\| \\
&\leq c2^{\mu\delta}\|f|F_{p,q}^s(E)\|
\end{aligned} \tag{91}$$

for a suitable $\delta > 0$, where $\widehat{\sup}$ is the supremum over

$$\left\{(y,t) \in \mathbb{R}^{n+1} : |x-y| \leq d't, \frac{t}{d'} \leq \tau \leq d't\right\}$$

for fixed $x \in \mathbb{R}^n$ and c does not depend on μ. Please note that the estimate of the norms by proposition 2.13 and the transformation of variables $\nu \to \nu - \mu$ were applied in the last step of the chain of proof. Hence we have to assume $p > \frac{n}{n+1}$.

Is it neglectable that the suprema in the second last term are taken over a small area of t resp. a larger area of t: This is obvious for the second part because we could extend the choosen domain $t \leq \tau \leq 2t$ in the proof of proposition 2.7 without any problems. For the first part this follows by the Taylor expansion of vector-valued functions

$$\|u(x,\tau)|E\| \leq c_k \sup \|\frac{\partial^k u(x,\tau)}{\partial t^k}|E\| + \sum_{l=0}^{k-1} c_l\|\frac{\partial^l u(x,1)}{\partial t^l}|E\|$$

(analogously for $\frac{\partial^l u(x,\tau)}{\partial t^l}$ instead of $u(x,\tau)$). Now one can estimate $\sup \| \frac{\partial^k u(x,\tau)}{\partial t^k} | E\|$ by the second term in (91).

We have to keep in mind that the constants of the equivalent norms depend on μ. But this can be incorporated by a factor $2^{\mu\delta}$. An analogous calculation shows the estimation of the norm $\|\lambda|b_{p,q}\|$ by $c\|f|B^s_{p,q}(E)\|$ in the case $f \in B^s_{p,q}(E)$.

Now we choose a $\psi \in \mathcal{S}(\mathbb{R}^n)$ with compact support and

$$\sum_{m\in\mathbb{Z}^n} \psi(x-m) = 1 \text{ for all } x \in \mathbb{R}^n. \tag{92}$$

For instance, one can choose a function $\psi_0 \in \mathcal{S}(\mathbb{R})$ with $\psi_0(x) = 0$ for $x \le -1$ and $\psi_0(x) = 1$ for $x \ge 1$. Then put $\psi(x_1,\dots,x_n) = (\psi_0(x_1) - \psi_0(x_1-1)) \cdot \ldots \cdot (\psi_0(x_n) - \psi_0(x_n-1))$. If we multiply (92) with $g \in \mathcal{S}(\mathbb{R}^n)$, then this converges in $\mathcal{S}(\mathbb{R}^n)$ - independent of the choice of ψ.

If $\nu > \mu$ and $m \in \mathbb{Z}^n$, we put (with c out of (87))

$$a_{\nu,m,l}(x) := c\lambda^{-1}_{\nu,m}\psi(2^\nu x - m) \int_{2^{-\nu+\mu}+l2^{-\nu}}^{2^{-\nu+\mu}+(l+1)2^{-\nu}} t^k \frac{\partial^k u(x,t)}{\partial t^k}\,\frac{dt}{t}$$

and

$$a_{\nu,m}(x) := \sum_{l=0}^{2^\mu-1} a_{\nu,m,l}(x).$$

In the case $\nu = \mu$ we define for $m \in \mathbb{Z}^n$

$$a_{\mu,m}(x) := \sum_{l=0}^{k-1} a_{\mu,m,l}(x)$$

with

$$a_{\mu,m,l}(x) := c^k_l \lambda^{-1}_{\mu,m}\psi(2^\mu x - m)\frac{\partial^l u(x,1)}{\partial t^l}.$$

Then we obtain (in $\mathcal{S}'(\mathbb{R}^n, E)$)

$$\begin{aligned}
f &= \sum_{l=0}^{k-1} c^k_l \frac{\partial^l u(\cdot,1)}{\partial t^l} + c\sum_{\nu=0}^{\infty}\int_{2^{-(\nu+1)}}^{2^{-\nu}} t^k\frac{\partial^k u(\cdot,t)}{\partial t^k}\,\frac{dt}{t} \\
&= \sum_{l=0}^{k-1} c^k_l \frac{\partial^l u(\cdot,1)}{\partial t^l} + c\sum_{\nu=\mu+1}^{\infty}\sum_{m\in\mathbb{Z}^n}\psi(2^\nu\cdot - m)\int_{2^{-\nu+\mu}}^{2^{-\nu+\mu+1}} t^k\frac{\partial^k u(\cdot,t)}{\partial t^k}\,\frac{dt}{t} \\
&= \sum_{\nu=\mu}^{\infty}\sum_{m\in\mathbb{Z}^n}\lambda_{\nu,m}a_{\nu,m}.
\end{aligned}$$

This is the desired representation. In the following we are going to show that the $a_{\nu,m}$ behave like E-valued $(s,p)_{K,-1}$ atoms for all $K \in \mathbb{N}$.

By construction the condition (74) is valid, more exactly it holds

$$\mathit{supp}\ a_{\nu,m} \subset d \cdot Q_{\nu,m}$$

for all $\nu \ge \mu$, $m \in \mathbb{Z}^n$ and a certain $d > 0$. We can't show any moment conditions (see (76)). To check the conditions on the derivatives we use a lemma for harmonic functions.

Lemma 3.6. *Let $W(X_1, \cdots, X_N) : \mathbb{R}^N \to E$ be harmonic in the domain*

$$K_R = \left\{X \in \mathbb{R}^N : |X| \leq R\right\}.$$

Then for $\varkappa \in (0,1)$ it is true that

$$\|D^\alpha W(X)|E\| \leq c_{\alpha,\varkappa} R^{-|\alpha|} \sup_{|Y|=R} \|W(Y)|E\| \text{ for } |X| \leq \varkappa R$$

with a constant c which depends on α and $\varkappa$ but not on R.

Proof. If $V : \mathbb{R}^N \to \mathbb{R}$ is harmonic in the given domain, then it holds

$$V(X) = \frac{R^2 - |X|^2}{R\omega_N} \int_{|Y|=R} \frac{V(Y)}{|X-Y|^N} \, ds_Y \text{ for } |X| < R, \tag{93}$$

where ω_N is the volume of the unit ball of $\mathbb{R}^N$. This relation follows from the formula of Dirichlet for the solution of the boundary value problem $\Delta g = 0$ in the domain $\left\{X \in \mathbb{R}^N : |X| < R\right\}$ and $g = f$ on $\left\{X \in \mathbb{R}^N : |X| = R\right\}$ and the uniqueness of the solution which is a consequence of the maximum principle, see [StW90], chapter 2, theorem 1.1, corollary 1.4 and theorem 1.10.

If $W : \mathbb{R}^N \to E$ is harmonic in such a domain, then the function $a(W) : \mathbb{R}^N \to \mathbb{R}$ - defined for an $a \in E'$ by $a(W)(x) := a(W(x))$ - is harmonic as well in this domain as one can easily recalculate. Therefore, formula (93) is valid and we obtain

$$\begin{aligned} a(W)(X) &= \frac{R^2 - |X|^2}{R\omega_N} \int_{|Y|=R} \frac{a(W)(Y)}{|X-Y|^N} \, ds_Y \\ &= a\left(\frac{R^2 - |X|^2}{R\omega_N} \int_{|Y|=R} \frac{W(Y)}{|X-Y|^N} \, ds_Y\right) \end{aligned}$$

for $|X| < R$. Since this holds for all $a \in E'$, both functions have to coincide in every point and we get the E-valued version of (93). If we take the derivative of both sides and interchange it with the integration, use

$$D^\alpha(|X|^N) = p(X)|X|^{N-2|\alpha|},$$

where $p(X)$ is a polynomial in $X_1, \ldots, X_N$ with degree $|\alpha|$ at most, and

$$\begin{aligned} |X - Y| &\geq (1-\varkappa)R, \\ \frac{|X|}{R^2 - |X|^2} &\leq \frac{1}{(1-\varkappa^2)R}, \\ \frac{1}{R^2 - |X|^2} &\leq \frac{1}{(1-\varkappa^2)R^2} \end{aligned}$$

for $|Y| = R$ and $|X| \leq \varkappa R$, then we obtain the relation

$$\|D^\alpha W(X)|E\| \leq c_{\alpha,\varkappa} R^{-|\alpha|} \sup_{|Y|=R} \|W(Y)|E\|$$

for $|X| \leq \varkappa R$. □

Now we apply this lemma to the functions $W(X) = \frac{\partial^k u(x,t)}{\partial t^k}$, which are harmonic in $\mathbb{R}^{n+1}_+$, to the balls $B_{\nu,m,l}$ instead of K_R with $R = 2^{-\nu+\mu-2}$ and to $\varkappa = d' 2^{-\mu+2}$.[8] Thus we obtain for $\nu > \mu$ and the set of all

$$\{(x,t) \in \mathbb{R}^{n+1} : |(2^{-\nu}m - x, 2^{-\nu+\mu} + l2^{-\nu} - t)| \leq d' 2^{-\mu+2} \cdot 2^{-\nu+\mu-2} = d' 2^{-\nu}\}$$

the relation

$$\left\| D^\gamma \frac{\partial^k u(x,t)}{\partial t^k} | E \right\| \leq c\, 2^{(\nu-\mu+2)|\gamma|} \sup \left\| \frac{\partial^k u(x,t)}{\partial t^k} | E \right\|,$$

where the supremum is taken over

$$\{(x,t) \in \mathbb{R}^{n+1} : |(2^{-\nu}m - x, 2^{-\nu+\mu} + l2^{-\nu} - t)| = 2^{-\nu+\mu-2}\}.$$

But for all $l \in \{0, \ldots, 2^\mu - 1\}$ this set is contained in the set $(x,t) \in \mathbb{R}^{n+1}$ with $|2^{-\nu}m - x| \leq d 2^{-\nu+\mu-1}$ and $d^{-1} 2^{-\nu+\mu} \leq t \leq d 2^{-\nu+\mu+1}$ for a suitable d. Having in mind the definition of $\lambda_{\nu,m}$ in (90), the support of $\psi(2^\nu \cdot - m)$ and the interval over which we integrate with respect to t we obtain

$$\|D^\gamma a_{\nu,m} | E\| \leq c \sum_{l=0}^{2^\mu - 1} 2^{-\nu\left(s - \frac{n}{p}\right) + \nu|\gamma|} 2^{\nu k} \int_{2^{-\nu+\mu} + l2^{-\nu}}^{2^{-\nu+\mu} + (l+1)2^{-\nu}} t^k \, \frac{dt}{t} \leq c' 2^{\mu k} 2^{-\nu(s - \frac{n}{p}) + \nu|\gamma|}.$$

Analogous assertions hold true for $a_{\mu,m}$ ($m \in \mathbb{Z}^n$). Therefore, we have proven the desired conditions (75) for all $K \in \mathbb{N}_0$. The $a_{\nu,m}$ introduced above for $\nu \in \mathbb{N}$, $\nu \geq \mu$ and $m \in \mathbb{Z}^n$ are E-valued $(s,p)_{K,-1}$-atoms for all $K \in \mathbb{N}_0$ - up to a constant which is independent of ν and m but depends on γ and μ. This constant can be transfered to the coefficients $\lambda_{\nu,m}$ for fixed K and μ.

We call this atoms "harmonic" because they are the product of a smooth function with a function which is derived from one which is harmonic in $\mathbb{R}^{n+1}_+$. Thus we call the found representation for f "harmonic" as well.

Proposition 3.7. *(i) Let $\frac{n}{n+1} < p \leq \infty$, $0 < q \leq \infty$, $s > \sigma_p$ and $K \in \mathbb{N}_0$ with $K \geq 1 + \lfloor s \rfloor$. Then $f \in \mathcal{S}'(\mathbb{R}^n, E)$ belongs to $B^s_{p,q}(E)$ if and only if it can be represented by*

$$f = \sum_{\nu \in \mathbb{N}_0} \sum_{m \in \mathbb{Z}^n} \lambda_{\nu,m} a_{\nu,m}(x).$$

Here $a_{\nu,m}$ are E-valued 1_K-atoms (for $\nu = 0$) or E-valued $(s,p)_{K,-1}$-atoms (for $\nu \in \mathbb{N}$) and $\lambda \in b_{p,q}$. Furthermore, we have

$$\|f | B^s_{p,q}(E)\| \sim \inf \|\lambda | b_{p,q}\|$$

in the sense of equivalence of norms, where the infimum on the right-hand side is taken over all admissible representations for f.

[8] It is $d' 2^{-\mu+2} < 1$ for $\mu > \varkappa_0$ and hence the calculations remain valid. Furthermore, one gets by a translation argument that lemma 3.6 is also true (in a slightly modified sense) for functions which are harmonic in a ball with center different from 0.

(ii) Let $\frac{n}{n+1} < p < \infty$, $0 < q \leq \infty$, $s > \sigma_{p,q}$ *and* $K \in \mathbb{N}_0$ *with* $K \geq 1 + \lfloor s \rfloor$. *Then* $f \in \mathcal{S}'(\mathbb{R}^n, E)$ *belongs to* $F^s_{p,q}(E)$ *if and only if it can be represented by*

$$f = \sum_{\nu \in \mathbb{N}_0} \sum_{m \in \mathbb{Z}^n} \lambda_{\nu,m} a_{\nu,m}(x).$$

Here $a_{\nu,m}$ *are* E*-valued* 1_K*-atoms (for* $\nu = 0$*) or* E*-valued* $(s,p)_{K,-1}$*-atoms (for* $\nu \in \mathbb{N}$*) and* $\lambda \in f_{p,q}$. *Furthermore, we have*

$$\|f|F^s_{p,q}(E)\| \sim \inf \|\lambda|f_{p,q}\|$$

in the sense of equivalence of norms, where the infimum on the right-hand side is taken over all admissible representations for f.

Proof. The assertions follow from 3.4 because the choice of $L = -1$ is admissible in the case $s > \sigma_p$ resp. $s > \sigma_{p,q}$ and from the proven representation observing (91) resp. the analogous result for $B^s_{p,q}(E)$. Here the coefficients $\lambda_{\nu,m}$ even vanish for $\nu < \mu$. We could demand additionally that the atoms are harmonic in the given sense. □

3.2 Subatomic decompositions

The aim of the following section will be to simplify the atomic representation of $f \in B^s_{p,q}(E)$ resp. $F^s_{p,q}(E)$ further. As a basis we use the harmonic representation from the last section. We orientate on [Tri97], section 14, p. 89 which treats the scalar case. At first we introduce a class of elementary functions which are special atoms - the γ-quarks.

Definition 3.8. Let $\psi \in \mathcal{S}(\mathbb{R}^n)$ with *supp* $\psi \subset d \cdot Q_{0,0}$ for a $d > 1$ and

$$\sum_{m \in \mathbb{Z}^n} \psi(x - m) = 1.$$

Let $s \in \mathbb{R}$, $0 < p \leq \infty$, $\frac{L+1}{2} \in \mathbb{N}_0$ and $\gamma \in \mathbb{N}^n_0$. We put $\psi^\gamma(x) := x^\gamma \psi(x)$. Then we call

$$(\gamma qu)^L_{\nu,m}(x) = 2^{-\nu\left(s - \frac{n}{p}\right)} \left((-\Delta)^{\frac{L+1}{2}} \psi^\gamma\right)(2^\nu x - m) \tag{94}$$

a $(s,p)_L$-γ-quark for $Q_{\nu,m}$. If $L = -1$, we want to denote it shortly by $(\gamma qu)_{\nu,m}(x)$.

Remark 3.9. First of all, we want to show that the $(s,p)_L - \gamma$-quarks really are (scalar) $(s,p)_{K,L}$-atoms for all $K \in \mathbb{N}_0$. The moment conditions (76) easily follow from their shape. For the derivatives we have

$$\begin{aligned}\left| D^\alpha \left(\left((-\Delta)^{\frac{L+1}{2}} \psi^\gamma \right)(2^\nu x - m) \right) \right| &\leq c 2^{|\alpha|\nu} \sum_{\alpha_1 + \alpha_2 = \alpha + L + 1} \left| \left(D^{\alpha_1} x^\gamma D^{\alpha_2} \psi \right)(2^\nu x - m) \right| \\ &\leq c\, 2^{|\alpha|\nu},\end{aligned}$$

where c depends on α and γ but not on ν or m. Hence $(\gamma qu)^L_{\mu,m}(x)$ is a $(s,p)_{K,L}$-atom if we normalize it in a suitable way.

Now we will simplify the representation of $f \in B^s_{p,q}(E)$ resp. $F^s_{p,q}(E)$ by the following result which corresponds to (in the scalar case) [Tri97], theorem 14.4, step 2, p. 93.

Lemma 3.10. *(i) Let* $\frac{n}{n+1} < p \le \infty$, $0 < q \le \infty$, $s > \sigma_p$ *and* $f \in B^s_{p,q}(E)$*. Then there is a* $\varkappa_0 \in \mathbb{N}$ *such that there exists a representation*

$$f = \sum_{\gamma \in \mathbb{N}_0^n} \sum_{\nu=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda^\gamma_{\nu,m} e^\gamma_{\nu,m} (\gamma qu)_{\nu,m}(x)$$

in $\mathcal{S}'(\mathbb{R}^n, E)$ *with* $e^\gamma_{\nu,m} \in U_E$ *for all* $\mu \ge \varkappa_0$*. It holds*

$$\sup_{\gamma \in \mathbb{N}_0^n} 2^{\mu|\gamma|} \|\lambda^\gamma | b_{p,q}\| \le c \|f|B^s_{p,q}(E)\|,$$

where c *does not depend on* f *and* $\lambda^\gamma = (\lambda^\gamma_{\nu,m})_{m \in \mathbb{Z}^n, \nu \in \mathbb{N}_0}$.

(ii) Let $\frac{n}{n+1} < p < \infty$, $0 < q \le \infty$, $s > \sigma_{p,q}$ *and* $f \in F^s_{p,q}(E)$*. Then there is a* $\varkappa_0 \in \mathbb{N}$ *such that there exists a representation*

$$f = \sum_{\gamma \in \mathbb{N}_0^n} \sum_{\nu=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda^\gamma_{\nu,m} (\gamma qu)_{\nu,m}(x) e^\gamma_{\nu,m},$$

in $\mathcal{S}'(\mathbb{R}^n, E)$ *with* $e^\gamma_{\nu,m} \in U_E$ *for all* $\mu \ge \varkappa_0$*. It holds*

$$\sup_{\gamma \in \mathbb{N}_0^n} 2^{\mu|\gamma|} \|\lambda^\gamma | f_{p,q}\| \le c \|f|F^s_{p,q}(E)\|,$$

where c *does not depend on* f *and* $\lambda^\gamma = (\lambda^\gamma_{\nu,m})_{m \in \mathbb{Z}^n, \nu \in \mathbb{N}_0}$.

Proof. We restrict ourselves to the case $f \in F^s_{p,q}(E)$, the case $f \in B^s_{p,q}(E)$ can be proven analogously. From proposition 3.7 and the previous remarks we obtain the optimal decomposition

$$f = \sum_{\nu \ge \mu} \sum_{m \in \mathbb{Z}^n} \lambda_{\nu,m} a_{\nu,m} \tag{95}$$

with (in case of $\nu > \mu$)

$$\begin{aligned} a_{\nu,m,l}(x) &= c\lambda^{-1}_{\nu,m} \psi(2^\nu x - m) \int_{2^{-\nu+\mu}+l2^{-\nu}}^{2^{-\nu+\mu}+(l+1)2^{-\nu}} t^k \frac{\partial^k u(x,t)}{\partial t^k} \frac{dt}{t}, \\ a_{\nu,m}(x) &= \sum_{l=0}^{2^\mu - 1} a_{\nu,m,l}(x), \\ \|\lambda | f_{p,q}\| &\le c \|f|F^s_{p,q}(E)\|. \end{aligned} \tag{96}$$

We want to expand the arbitrarily often differentiable functions $\frac{\partial^k u(x,t)}{\partial t^k}$ into a Taylor series, with center $(2^{-\nu}m, 2^{-\nu+\mu} + l2^{-v})$ of the balls $B_{\nu,m,l}$. We need a result for harmonic functions for a further estimate.

Lemma 3.11. *There exist $c > 0$ and $0 < \tau < 1$ such that*

$$\|D^\alpha W(0)|E\| \le c\alpha!\tau^{-|\alpha|} \sup_{|y|=1} \|W(y)|E\|$$

for all $\alpha \in \mathbb{N}_0^N$ and all $W : \mathbb{R}^N \to E$ which are harmonic in the domain $\{y \in \mathbb{R}^N : |y| \le 1\}$.

Proof. For an arbitrary $Y \in \mathbb{R}^N$ with $|Y| = 1$ we expand the function

$$|Z - Y|^{-N} = \left[\sum_{j=1}^N (Z_j - Y_j)^2\right]^{-\frac{N}{2}},$$

which is holomorphic in $|Z| < c$,[9] in its Taylor series around 0. At this c does not depend on Y. By a repeated application of the Cauchy formula for $\tau \le c' < \frac{c}{\sqrt{N}}$ it follows

$$\begin{aligned} &\left|\left(D^\alpha |Z - Y|^{-N}\right)(0)\right| \\ &\quad = \left|(2\pi i)^{-N}(-1)^\alpha \alpha! \int_{|z_1|=\tau} \cdots \int_{|z_N|=\tau} \frac{|(z_1, \ldots, z_N) - Y|^{-N}}{z_1^{\alpha_1+1} \cdot \ldots \cdot z_N^{\alpha_N+1}}\, dz_1 \ldots dz_N\right| \\ &\quad \le c\tau^{-|\alpha|}\alpha! \end{aligned}$$

uniformly in $Y \in \mathbb{R}^N$ with $|Y| = 1$.

By the formula of Dirichlet for E-valued functions which are harmonic in $\{Y \in \mathbb{R}^N : |Y| \le 1\}$ (see (93)) and by the uniform convergence of the Taylor series of $D^\alpha |Z - Y|^{-N}$ for $X, Y \in \mathbb{R}^N$ with $|Y| = 1$ and $|X| < \tau$ we obtain

$$\begin{aligned} W(X) &= \frac{1 - |X|^2}{\omega_N} \int_{|Y|=1} \frac{W(Y)}{|X - Y|^N}\, ds_Y \\ &= \frac{1 - |X|^2}{\omega_N} \int_{|Y|=1} \sum_{\alpha \in \mathbb{N}_0^n} \frac{X^\alpha}{\alpha!} a_\alpha(Y) W(Y)\, ds_Y \\ &= \sum_{\alpha \in \mathbb{N}_0^N} \frac{X^\alpha}{\omega_N \alpha!} \int_{|Y|=1} \tilde{a}_\alpha(Y) W(Y)\, ds_Y \end{aligned} \tag{97}$$

with $a_\alpha(Y) = D^\alpha \left(|Z - Y|^{-N}\right)(0)$ and

$$\tilde{a}_\alpha(y) = a_\alpha(y) - \sum_{k=1}^N a_{\alpha - 2e_k}(y),$$

[9] Please note that the root function is holomorphic in all $z \in \mathbb{C} \setminus (\mathbb{R}^- \cup \{0\})$ and that for all $Y \in \mathbb{R}^N$ with $|Y| = 1$ and all $\varepsilon > 0$ it is valid that

$$\left|\sum_{j=1}^N (Z_j - Y_j)^2 - 1\right| < \varepsilon$$

if we choose $|Z|$ small enough.

where e_k is the multi-index $(0,\dots,0,1,0,\dots,0)$ which is 1 at the k-th position and elsewhere 0 and where coefficients for negative multi-indices should be 0. By taking the derivative of the power series we see that

$$\|(D^\alpha W)(0)|E\| = \frac{1}{\omega_N}\left\|\int_{|Y|=1} \tilde{a}_\alpha(Y)W(Y)\ ds_Y\Big|E\right\| \le c\alpha!\tau^{-|\alpha|}\sup_{|y|=1}\|W(y)|E\|$$

is valid for all $\alpha \in \mathbb{N}_0^N$, where c does not depend on α and W. □

Now we apply this lemma with $N = n+1$ to the functions $W(x,t) := \frac{\partial^k u(x,t)}{\partial t^k}$ which are harmonic in the ball $B_{\nu,m,l}$. If we set

$$\tilde{W}(x,t) := W\left(2^{-\nu+\mu-2}x + 2^{-\nu}m, 2^{-\nu+\mu-2}t + 2^{-\nu+\mu} + l2^{-\nu}\right),$$

then $\tilde{W}$ is harmonic in $\{y \in \mathbb{R}^{n+1} : |y| \le 1\}$ and we obtain

$$\|(D^\alpha \tilde{W})(0)|E\| \le c\alpha!\tau^{-|\alpha|}\sup_{|y|=1}\|\tilde{W}(y)|E\|,$$

where c is independent of k, ν, μ, l and α, and a representation as in (97) for $\tilde{W}(x,t)$ if $|(x,t)| < \tau$. Hence we finally get the power series expansion

$$\begin{aligned} W(x,t) &= \tilde{W}(2^{\nu-\mu+2}x - 2^{-\mu+2}m, 2^{\nu-\mu+2}t - 2^2 - l2^{-\mu+2}) \\ &= \sum_{\alpha\in\mathbb{N}_0^N,\ \beta\in\mathbb{N}_0} c_{(\alpha,\beta)}2^{(|\alpha|+\beta)(\nu-\mu+2)}\frac{(x-2^{-\nu}m)^\alpha \cdot (t-2^{-\nu+\mu}-l2^{-\nu})^\beta}{\alpha!\beta!} \end{aligned} \qquad (98)$$

for $|(2^{\nu-\mu+2}x - 2^{-\mu+2}m, 2^{\nu-\mu+2}t - 2^2 - l2^{-\mu+2})| < \tau$, i.e. for

$$|(x-2^{-\nu}m, t-2^{-\nu+\mu}-l2^{-\nu})| < 2^{-\nu+\mu-2}\tau.$$

If we choose μ larger or equal to a certain $\varkappa_0$, then this expansion is true in particular for $(x,t) \in \mathbb{R}^{n+1}$ with $x \in \mathit{supp}\ \psi(2^\nu x - m)$ and $t \in [2^{-\nu+\mu}+l2^{-\nu}, 2^{-\nu+\mu}+(l+1)2^{-\nu}]$. Here we have

$$\|c_{\alpha,\beta}|E\| = \|(D^{\alpha,\beta}\tilde{W}(0)|E\| \le c\alpha!\tau^{-|\alpha|-\beta|}\sup_{(x,t)\in B_{\nu,m,l}}\|W(y)|E\|$$

by lemma 3.11 proven before. If we put this into (96), we obtain

$$\begin{aligned} a_{\nu,m,l}(x) &= c\lambda_{\nu,m}^{-1}\psi(2^\nu x - m)\int_{2^{-\nu+\mu}+l2^{-\nu}}^{2^{-\nu+\mu}+(l+1)2^{-\nu}} t^k\frac{\partial^k u(x,t)}{\partial t^k}\ \frac{dt}{t} \\ &= c\lambda_{\nu,m}^{-1}\psi(2^\nu x - m)\sum_{\alpha\in\mathbb{N}_0^N,\ \beta\in\mathbb{N}_0} c_{(\alpha,\beta)}2^{(|\alpha|+\beta)(\nu-\mu+2)}\frac{(x-2^{-\nu}m)^\alpha}{\alpha!\beta!} \\ &\quad\cdot\int_{2^{-\nu+\mu}+l2^{-\nu}}^{2^{-\nu+\mu}+(l+1)2^{-\nu}} (t-2^{-\nu+\mu}-l2^{-\nu})^\beta t^k\ \frac{dt}{t} \\ &\equiv \sum_{\gamma\in\mathbb{N}_0^N} \tilde{c}_\gamma\ (2^\nu x - m)^\gamma\psi(2^\nu x - m) \end{aligned}$$

with

$$\begin{aligned}\|\tilde{c}_\gamma|E\| &\le c2^{-\nu|\gamma|}\lambda_{\nu,m}^{-1}\sum_{\beta=0}^{\infty}\frac{2^{(\nu-\mu+2)(|\gamma|+\beta)}}{\gamma!\beta!}\ c_{(\gamma,\beta)}\int_{2^{-\nu+\mu}+l2^{-\nu}}^{2^{-\nu+\mu}+(l+1)2^{-\nu}}(t-2^{-\nu+\mu}-l2^{-\nu})^\beta t^k\ \frac{dt}{t} \\ &\le c'2^{-\nu\left(s-\frac{n}{p}\right)}2^{\nu k}2^{-\nu|\gamma|}\sum_{\beta=0}^{\infty}2^{(\nu-\mu+2)(|\gamma|+\beta)}\tau^{-|\gamma|-\beta}2^{-\nu\beta+(\mu-\nu)k} \\ &= c'2^{-\nu\left(s-\frac{n}{p}\right)}2^{\mu k}2^{(-\mu+2)|\gamma|}\tau^{-|\gamma|}\sum_{\beta=0}^{\infty}\tau^{-\beta}2^{(-\mu+2)\beta}\end{aligned}$$

observing the definition of $\lambda_{\nu,m}$ in (90). By our choice of $\varkappa_0$ we have $\tau^{-1}2^{-\mu+2}<1$ for $\mu\ge\varkappa_0$. Hence the series over β converges and (keeping in mind the definition of the γ-quarks from (94) for $L=-1$) we obtain

$$a_{\nu,m,l}(x)=\sum_{\gamma\in\mathbb{N}_0^n}\eta_{\nu,m,l}^{\gamma}(\gamma qu)_{\nu,m}(x)$$

with

$$\|\eta_{\nu,m,l}^{\gamma}|E\|\le c''2^{\mu k}(\tau^{-1}2^{-\mu+2})^{|\gamma|}.$$

Here c'' and τ are independent of μ and γ. If we replace μ by $M\mu$ afterwards, where $M\in\mathbb{N}$ is sufficiently large, and sum over $l=0,\ldots,2^\mu-1$ in (96), we arrive at

$$a_{\nu,m}(x)=\sum_{\gamma\in\mathbb{N}_0^n}\eta_{\nu,m}^{\gamma}(\gamma qu)_{\nu,m}(x)$$

with

$$\|\eta_{\nu,m}^{\gamma}|E\|\le C2^{\mu\delta}2^{-\mu|\gamma|}$$

for certain $C>0$ and $\delta>0$ which do not depend on μ and γ.

The case $\nu=\mu$ can be treated analogously. One just has to observe that (98) with $\nu=\mu$ and $l=0$ holds on a set (nearly) independent of μ. Hence the support of $\psi(2^\mu x-m)$ for $\mu\ge\varkappa_0$ is contained in this set. Furthermore, one has to set $t=1$ in the Taylor expansion so that the sum over β in (98) vanishes.

Hence we obtain from (95)

$$f=\sum_{\gamma\in\mathbb{N}_0^n}\sum_{\nu=0}^{\infty}\sum_{m\in\mathbb{Z}^n}\lambda_{\nu,m}^{\gamma}(\gamma qu)_{\nu,m}(x)e_{\nu,m}^{\gamma}$$

in $\mathcal{S}'(\mathbb{R}^n,E)$ with

$$\lambda_{\nu,m}^{\gamma}=\begin{cases}\lambda_{\nu,m}\|\eta_{\nu,m}^{\gamma}|E\| & ,\nu\ge\mu\\ 0 & ,\nu<\mu\end{cases}$$

and

$$e_{\nu,m}^{\gamma}=\begin{cases}\frac{\eta_{\nu,m}^{\gamma}}{\|\eta_{\nu,m}^{\gamma}|E\|} & ,\nu\ge\mu\\ 0 & ,\nu<\mu\end{cases}.$$

With (91) and the observations on the dependency of μ we come to

$$2^{\mu|\gamma|}\|\lambda^\gamma|f_{p,q}\| \le C' 2^{\mu\delta_1}\|f|F^s_{p,q}(E)\|, \quad \gamma \in \mathbb{N}_0^n$$

with C' and δ_1 independent of μ and γ for $\mu \ge \varkappa_0$. □

Now we have all the ingredients together to prove [Tri97], theorem 15.8, p. 114, where now arbitrary $s \in \mathbb{R}$ are allowed.

Theorem 3.12. *(i) Let $\frac{n}{n+1} < p \le \infty$, $0 < q \le \infty$ and $s \in \mathbb{R}$. Let $M \in \mathbb{N}$ with $M > \sigma_p$ and $M > s$ and L with $\frac{L+1}{2} \in \mathbb{N}_0$ and $L \ge \lfloor \sigma_p - s \rfloor$ be fixed. Let $(\gamma qu)_{\nu,m}$ and $(\gamma qu)^L_{\nu,m}$ be given as $(M,p)_{-1}$- resp. $(s,p)_L$-γ-quarks for a given function $\psi \in \mathcal{S}(\mathbb{R}^n)$ with compact support and the property (92). Then there exists a $\varkappa > 0$ such that for all $\mu \ge \varkappa$ it is valid that $f \in \mathcal{S}'(\mathbb{R}^n, E)$ belongs to $B^s_{p,q}(E)$ if and only if it can be represented as*

$$f = \sum_{\gamma\in\mathbb{N}_0^n}\sum_{\nu=0}^{\infty}\sum_{m\in\mathbb{Z}^n} \varrho^\gamma_{\nu,m} e^\gamma_{\nu,m}(\gamma qu)_{\nu,m}(x) + \lambda^\gamma_{\nu,m} e^{\gamma,L}_{\nu,m}(\gamma qu)^L_{\nu,m}(x)$$

in $\mathcal{S}'(\mathbb{R}^n, E)$ with $e^\gamma_{\nu,m}, e^{\gamma,L}_{\nu,m} \in U_E$ and

$$\sup_{\gamma\in\mathbb{N}_0} 2^{\mu|\gamma|}(\|\varrho^\gamma|b_{p,q}\| + \|\lambda^\gamma|b_{p,q}\|) < \infty.$$

Furthermore, it holds in the sense of equivalence of norms

$$\|f|B^s_{p,q}(E)\| \sim \inf \sup_{\gamma} 2^{\mu|\gamma|}(\|\varrho^\gamma|b_{p,q}\| + \|\lambda^\gamma|b_{p,q}\|),$$

where the infimum on the right-hand side is taken over all admissible representations of f.

(ii) Let $\frac{n}{n+1} < p < \infty$, $0 < q \le \infty$ and $s \in \mathbb{R}$. Let $M \in \mathbb{N}$ with $M > \sigma_{p,q}$ and $M > s$ and L with $\frac{L+1}{2} \in \mathbb{N}_0$ and $L \ge \lfloor \sigma_{p,q} - s \rfloor$ be given. The quarks have the same meaning as in (i). Then there exists a $\varkappa > 0$ such that for all $\mu \ge \varkappa$ it is valid that $f \in \mathcal{S}'(\mathbb{R}^n, E)$ belongs to $F^s_{p,q}(E)$ if and only if it can be represented as

$$f = \sum_{\gamma\in\mathbb{N}_0^n}\sum_{\nu=0}^{\infty}\sum_{m\in\mathbb{Z}^n} \varrho^\gamma_{\nu,m} e^\gamma_{\nu,m}(\gamma qu)_{\nu,m}(x) + \lambda^\gamma_{\nu,m} e^{\gamma,L}_{\nu,m}(\gamma qu)^L_{\nu,m}(x) \tag{99}$$

in $\mathcal{S}'(\mathbb{R}^n, E)$ with $e^\gamma_{\nu,m}, e^{\gamma,L}_{\nu,m} \in U_E$ and

$$\sup_{\gamma} 2^{\mu|\gamma|}(\|\varrho^\gamma|f_{p,q}\| + \|\lambda^\gamma|f_{p,q}\|) < \infty. \tag{100}$$

Furthermore, it holds in the sense of equivalence of norms

$$\|f|F^s_{p,q}(E)\| \sim \inf \sup_{\gamma\in\mathbb{N}_0} 2^{\mu|\gamma|}(\|\varrho^\gamma|f_{p,q}\| + \|\lambda^\gamma|f_{p,q}\|),$$

where the infimum on the right-hand side is taken over all admissible representations of f.

Proof. We only consider the case $F^s_{p,q}(E)$. The proof for $B^s_{p,q}(E)$ can be organized analogously. Let $f \in \mathcal{S}'(\mathbb{R}^n, E)$ be represented by (99) with the condition (100). Then

$$f_1^\gamma := \sum_{\nu=0}^{\infty} \sum_{m\in\mathbb{Z}^n} \varrho^\gamma_{\nu,m} e^\gamma_{\nu,m} (\gamma qu)_{\nu,m}(x)$$

and

$$f_2^\gamma := \sum_{\nu=0}^{\infty} \sum_{m\in\mathbb{Z}^n} \lambda^\gamma_{\nu,m} e^{\gamma,L}_{\nu,m} (\gamma qu)^L_{\nu,m}(x)$$

are represented as sums of atoms (up to a constant) as elements of $\mathcal{S}'(\mathbb{R}^n, E)$ by remark 3.9. Here $(\gamma qu)_{\nu,m} e^\gamma_{\nu,m}$ resp. $(\gamma qu)^L_{\nu,m} e^{\gamma,L}_{\nu,m}$ are E-valued $(M,p)_{K,-1}$- resp. $(s,p)_{K,L}$-atoms for every $K \in \mathbb{N}_0$, where one has to keep in mind a normalization constant $c2^{\varkappa\gamma}$ with c depending on K but independent of γ (see remark 3.9). Thus we obtain by proposition 3.4 that $f_1^\gamma \in F^M_{p,q}(E)$[10], that $f_2^\gamma \in F^s_{p,q}(E)$ and that there exists a $c'' > 0$ such that it holds

$$\|f^\gamma | F^s_{p,q}(E)\| \le c' \left(\|f_1^\gamma | F^M_{p,q}(E)\| + \|f_2^\gamma | F^s_{p,q}(E)\| \right) \le c'' 2^{\varkappa\gamma} \left(\|\varrho^\gamma | f_{p,q}\| + \|\lambda^\gamma | f_{p,q}\| \right).$$

with $f^\gamma = f_1^\gamma + f_2^\gamma$. Here c'' and $\varkappa$ are independent of γ (and f^γ). Therefore, if we take $\mu > \varkappa$ for granted, it results from (100) that

$$f = \sum_{\gamma\in\mathbb{N}_0^n} f^\gamma$$

in $F^s_{p,q}(E)$ and

$$\|f | F^s_{p,q}(E)\| \le C \sup_\gamma 2^{\mu|\gamma|} \left(\|\varrho^\gamma | f_{p,q}\| + \|\lambda^\gamma | f_{p,q}\| \right),$$

which follows in the case $p, q \ge 1$ from the triangle inequality and in the remaining cases (for $r \le \min(1,p,q)$ and $\varepsilon > 0$) from

$$\begin{aligned} \|g | L_p(l_q)\| &= \left(\|g^r | L_{\frac{p}{r}}(l_{\frac{q}{r}})\| \right)^{\frac{1}{r}} \le \left(\sum_{j=0}^{\infty} \|g_j^r | L_{\frac{p}{r}}(l_{\frac{q}{r}})\| \right)^{\frac{1}{r}} \\ &= \left(\sum_{j=0}^{\infty} \|g_j | L_p(l_q)\|^r \right)^{\frac{1}{r}} \le c \sup_j 2^{j\varepsilon} \|g_j | L_p(l_q)\| \end{aligned}$$

for

$$g = \sum_{j=0}^{\infty} g_j.$$

Hence this part of the proof is even valid for all $0 < p \le \infty$, $0 < q \le \infty$ and $s \in \mathbb{R}$.

[10] Because of $M > \sigma_{p,q}$ we need no moment conditions (76) for these atoms.

Let f from $F^s_{p,q}(E)$ be given for the second part of the proof. In the case $s > \sigma_{p,q}$ and $L = -1$ the assertion of the proposition follows from lemma 3.10. Here we don't need any terms of the form $\varrho^\gamma_{\nu,m} e^\gamma_{\nu,m} (\gamma qu)_{\nu,m}(x)$.

Let now s be arbitrary and $f \in F^s_{p,q}(E)$. Then we have by the lift property (see (17))

$$g = \left((1 + |\cdot|^2)^{-\frac{L+1}{2}} \hat{f} \right)^\vee \in F^{s+L+1}_{p,q}(E)$$

with $\|f|F^s_{p,q}(E)\| \sim \|g|F^{s+L+1}_{p,q}(E)\|$. Thus f can be represented as

$$f = g + (-\Delta)^{\frac{L+1}{2}} g.$$

If we apply the same argument to g, we obtain

$$f = h + (-\Delta)^{\frac{L+1}{2}} h + (-\Delta)^{\frac{L+1}{2}} g = h + (-\Delta)^{\frac{L+1}{2}} (g + h)$$

with $\|f|F^s_{p,q}(E)\| \sim \|g|F^{s+L+1}_{p,q}(E)\| + \|h|F^{s+2(L+1)}_{p,q}(E)\|$. An iteration argument eventually yields

$$f = f_1 + (-\Delta)^{\frac{L+1}{2}} f_2$$

with $\|f|F^s_{p,q}(E)\| \sim \|f_1|F^{s+m(L+1)}_{p,q}(E)\| + \|f_2|F^{s+L+1}_{p,q}(E)\|$. If $L \geq \lfloor \sigma_{p,q} - s \rfloor$ (and $\frac{L+1}{2} \in \mathbb{N}_0$), then $s + L + 1$ fulfils the conditions from lemma 3.10, this means $s + L + 1 > \sigma_{p,q}$. Then f_2 can be represented by

$$f_2 = \sum_{\gamma \in \mathbb{N}^n_0} \sum_{\nu=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda^\gamma_{\nu,m} e^{\gamma,L}_{\nu,m} (\gamma qu)_{\nu,m}(x),$$

where $(\gamma qu)_{\nu,m}$ are $(s + L + 1, p)_{-1}$-γ-quarks and it holds

$$\|f_2|F^{s+L+1}_{p,q}(E)\| \sim \sup_\gamma 2^{\mu|\gamma|} \|\lambda^\gamma|f_{p,q}\|.$$

But now we have

$$\begin{aligned} (-\Delta)^{\frac{L+1}{2}} (\gamma qu)_{\nu,m}(x) &= (-\Delta)^{\frac{L+1}{2}} \left(2^{-\nu\left(s+L+1-\frac{n}{p}\right)} \psi^\gamma(2^\nu x - m) \right) \\ &= 2^{-\nu\left(s-\frac{n}{p}\right)} \left((-\Delta)^{-\frac{L+1}{2}} \psi^\gamma \right) (2^\nu x - m), \end{aligned}$$

which is an $(s, p)_L$-γ-quark.

Furthermore, let's choose m so large that $\tilde{M} := s + m(L + 1)$ fulfils the condition $\tilde{M} \geq M$. From $f_1 \in F^{\tilde{M}}_{p,q}(E)$ follows $f_1 \in F^M_{p,q}(E)$ as well. This yields a representation for f_1 with $(M, p)_{-1}$-γ-quarks by lemma 3.10, observing $M > \sigma_{p,q}$. Hence we obtain a representation for f as a sum of $(M, p)_{-1}$- and $(s, p)_L$-γ-quarks by

$$f = \sum_{\gamma \in \mathbb{N}^n_0} \sum_{\nu=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \varrho^\gamma_{\nu,m} e^\gamma_{\nu,m} (\gamma qu)_{\nu,m}(x) + \lambda^\gamma_{\nu,m} e^{\gamma,L}_{\nu,m} (\gamma qu)^L_{\nu,m}(x)$$

and it holds by the previous steps

$$\begin{aligned}\sup_\gamma 2^{\mu|\gamma|}\left(\|\varrho^\gamma|f_{p,q}\|+\|\lambda^\gamma|f_{p,q}\|\right) &\le c\left(\|f_1|F^M_{p,q}(E)\|+\|f_2|F^{s+L+1}_{p,q}(E)\|\right)\\ &\le c\left(\|f_1|F^{\tilde M}_{p,q}(E)\|+\|f_2|F^{s+L+1}_{p,q}(E)\|\right)\\ &\le c'\|f|F^s_{p,q}(E)\|.\end{aligned}$$

□

Now it is an easy task to expand this theorem to the more general atoms. This was suggested by the first step of the preceding proof in which we only used that the quarks are atoms. We now obtain [Tri97], theorem 15.11, p. 116.

Theorem 3.13. *(i) Let $\frac{n}{n+1} < p \le \infty$, $0 < q \le \infty$ and $s \in \mathbb{R}$. Let $M \in \mathbb{N}$ with $M > \sigma_p$ and $M > s$, $K \in \mathbb{N}_0$ with $K \ge \lfloor s \rfloor + 1$ and L with $\frac{L+1}{2} \in \mathbb{N}_0$ and $L \ge \lfloor \sigma_p - s \rfloor$ be fixed. Then there exists a $\varkappa > 0$ such that for all $\mu \ge \varkappa$ it is valid that $f \in \mathcal{S}'(\mathbb{R}^n, E)$ belongs to $B^s_{p,q}(E)$ if and only if it can be represented by*

$$f = \sum_{\gamma\in\mathbb{N}_0^n}\sum_{\nu=0}^{\infty}\sum_{m\in\mathbb{Z}^n} \varrho^\gamma_{\nu,m} e^\gamma_{\nu,m} a^\gamma_{\nu,m}(x) + \lambda^\gamma_{\nu,m} e^{\gamma,L}_{\nu,m} a^{\gamma,L}_{\nu,m}(x)$$

in $\mathcal{S}'(\mathbb{R}^n, E)$. Here $a^\gamma_{\nu,m}$ resp. $a^{\gamma,L}_{\nu,m}$ are $(M,p)_{K,-1}$ resp. $(s,p)_{K,L}$-atoms, $e^\gamma_{\nu,m}, e^{\gamma,L}_{\nu,m} \in U_E$ and

$$\sup_\gamma 2^{\mu|\gamma|}(\|\varrho^\gamma|b_{p,q}\| + \|\lambda^\gamma|b_{p,q}\|) < \infty.$$

Furthermore, we have in the sense of equivalence of norms

$$\|f|B^s_{p,q}(E)\| \sim \inf\ \sup_\gamma 2^{\mu|\gamma|}(\|\varrho^\gamma|b_{p,q}\| + \|\lambda^\gamma|b_{p,q}\|),$$

where the infimum on the right-hand side is taken over all admissible representations of f.

(ii) Let $\frac{n}{n+1} < p < \infty$, $0 < q \le \infty$ and $s \in \mathbb{R}$. Let $M \in \mathbb{N}$ with $M > \sigma_{p,q}$ and $M > s$, $K \in \mathbb{N}_0$ with $K \ge \lfloor s \rfloor + 1$ and L with $\frac{L+1}{2} \in \mathbb{N}_0$ and $L \ge \lfloor \sigma_{p,q} - s \rfloor$ be fixed. Then there exists a $\varkappa > 0$ such that for all $\mu \ge \varkappa$ it is valid that $f \in \mathcal{S}'(\mathbb{R}^n, E)$ belongs to $F^s_{p,q}(E)$ if and only if it can be represented by

$$f = \sum_{\gamma\in\mathbb{N}_0^n}\sum_{\nu=0}^{\infty}\sum_{m\in\mathbb{Z}^n} \varrho^\gamma_{\nu,m} e^\gamma_{\nu,m} a^\gamma_{\nu,m}(x) + \lambda^\gamma_{\nu,m} e^{\gamma,L}_{\nu,m} a^{\gamma,L}_{\nu,m}(x)$$

in $\mathcal{S}'(\mathbb{R}^n, E)$. Here $a^\gamma_{\nu,m}$ resp. $a^{\gamma,L}_{\nu,m}$ are $(M,p)_{K,-1}$ resp. $(s,p)_{K,L}$-atoms, $e^\gamma_{\nu,m}, e^{\gamma,L}_{\nu,m} \in U_E$ and

$$\sup_\gamma 2^{\mu|\gamma|}(\|\varrho^\gamma|f_{p,q}\| + \|\lambda^\gamma|f_{p,q}\|) < \infty.$$

Furthermore, we have in the sense of equivalence of norms

$$\|f|F^s_{p,q}(E)\| \sim \inf\ \sup_\gamma 2^{\mu|\gamma|}(\|\varrho^\gamma|f_{p,q}\| + \|\lambda^\gamma|f_{p,q}\|),$$

where the infimum on the right-hand side is taken over all admissible representations of f.

Proof. The existence of such a representation for $f \in B^s_{p,q}(E)$ resp. $f \in F^s_{p,q}(E)$ follows from the fact that the $(s,p)_L$-γ-quarks are also $(s,p)_{K,L}$-atoms for all $K \in \mathbb{N}_0$ and by the previous theorem. If $f \in \mathcal{S}(\mathbb{R}^n, E)$ can be represented in the given way, then it belongs to $B^s_{p,q}(E)$ resp. $F^s_{p,q}(E)$ by the first step of the proof of the previous theorem because it only uses that the quarks are $(M,p)_{K,-1}$- resp. $(s,p)_{K,L}$-atoms. □

References

The bibliographical references are sorted by the names of the (first) authors.

[BPT96] Bui, H.-Q.; Paluszyński, M.; Taibleson, M.H.: A maximal function characterization of weighted Besov-Lipschitz and Triebel-Lizorkin spaces. Studia Math. 119 (1996), no.3, 219-246.

[BPT97] Bui, H.-Q.; Paluszyński, M.; Taibleson, M.H.: Characterization of the Besov-Lipschitz and Triebel-Lizorkin spaces. The case $q < 1$. J. of Fourier Analysis and Applications, vol. 3, special issue, 1997, 837-846.

[Gra04] Grafakos, L.: Classical and Modern Fourier Analysis. Pearson/ Prentice Hall, Upper Saddle River, NJ 2004.

[Ryc99] Rychkov, V. S.: On a Theorem of Bui, Paluszyński, and Taibleson. Proc. Steklov Inst. Math. 227, 280 (1999).

[ScS01] Schmeisser, H.-J.; Sickel, W.: Traces, Gagliardo-Nirenberg Inequalities and Sobolev Type Embeddings for Vector-valued Function Spaces. Jenaer Schriften zur Math. und Inf. (Mat/Inf/24/01), Jena 2001.

[StW90] Stein, E.M.; Weiss, G.: Introduction to Fourier Analysis on Euclidean Spaces. Princeton University Press, 6. Auflage, Princeton, NJ 1990

[Tri83] Triebel, H.: Theory of Function Spaces. Akademische Verlagsgesellschaft Geest & Portig, Leipzig 1983.

[Tri92] Triebel, H.: Theory of Function Spaces II. Birkhäuser, Basel 1992.

[Tri97] Triebel, H.: Fractals and Spectra. Birkhäuser, Basel 1997.

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Selbstständigkeitserklärung

Ich erkläre, dass ich die vorliegende Arbeit selbstständig und nur unter Verwendung der angegebenen Quellen und Hilfsmittel angefertigt habe.

Jena, 23. März 2009