Traces of vector-valued Sobolev Spaces

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We characterize the traces of vector-valued Besov and Lizorkin-Triebel spaces. Therefrom we derive the corresponding assertions for the vector-valued Sobolev spaces $W^m_p(\mathbf{R}^n, E)$. Here we do not assume the UMD property for the Banach space E.

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1 Introduction

The aim of the paper is to characterize the trace space of vector-valued Sobolev spaces $W^m_p(\mathbf{R}^n, E)$, where E is an arbitrary Banach space. In particular, we do not assume that the underlying Banach space E has the UMD property. Vector-valued Sobolev and Besov spaces are widely used in abstract evolution equations, cf. e.g. Amann [1, 2, 4], Veraar and Weis [57] or Denk, Hieber, Prüss, Saal, and Seiler [9, 10, 44, 11], and in the theory of integral operators, cf. e.g. König [31], Pietsch [41], and Hytönen and Veraar [25]. Let us also mention that traces for certain vector-valued Sobolev spaces on domains associated with parabolic problems have been investigated by P. Weidemaier [58, 59]. For certain applications it is quite inconvenient to assume that the Banach space E has the UMD property. There are simple examples of Banach spaces which do not have this property. The UMD property implies reflexivity of E. So it rules out Hölder spaces and L_1 and variants of those, cf. Amann [1]. On the other hand the UMD property is essential for the validity of Michlin-Hörmander type Fourier multiplier assertions in corresponding L_p-spaces. So, there is a battle in that field which properties of the distribution spaces really depend on the UMD property and which do not. Hence one has to avoid techniques which are based on such multipliers.

A way out is the use of vector-valued spaces of Besov- and Lizorkin-Triebel type, where multiplier theorems are available and their Fourier-analytic definition uses Littlewood-Paley type decompositions. These scales of spaces seem to be of interest also for themselves. However, in contrast to the scalar case they do not contain (fractional) Sobolev spaces as special cases. Nevertheless, it turns out that some assertions can be derived via rather elementary embeddings within the above scale of spaces. This concerns inequalities of Gagliardo-Nirenberg-type and related Sobolev type embeddings (see [50]) as well as limiting cases of embeddings (see [32]).

The paper is organized as follows. In Section 2 we introduce vector-valued Besov and Lizorkin-Triebel spaces. Section 3 deals with atomic and subatomic decompositions of vector-valued Besov and Lizorkin-Triebel spaces. These decompositions will be used as a main tool to prove our trace theorems in Section 4.

The paper is based on earlier manuscripts of the authors, see [49] and [47]. There one can find an extended treatment - in particular a characterization of Lizorkin-Triebel spaces by means of differences, characterizations

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by atoms, local means, limiting cases of embeddings and applications to spaces with dominating mixed smoothness.

2 Vector-valued Besov and Lizorkin-Triebel spaces

2.1 Preliminaries

Throughout the paper E denotes a Banach space and $\| \cdot \|_E$ its norm without any further restriction.

Some knowledge of the Bochner integral and the vector-valued Fourier transform $\mathcal{F}$ is assumed, see e.g. the monograph of Diestel and Uhl [12], the textbook of Amann and Escher [5], or Grafakos [23] for the first topic and [2, 3, 48] for the second. For basics of Lebesgue spaces $L_p(\mathbf{R}^n, E)$ we refer again to [5]. We fix some further notation. The set of all natural numbers will be denoted by $\mathbf{N}$, $\mathbf{Z}$ denotes the integers, $\mathbf{N}_0$ the nonnegative integers, $\mathbf{R}$ the reals and $\mathbf{R}^n$ the Euclidean n-space. The Schwartz space of all complex-valued infinitely differentiable and rapidly decreasing functions defined on $\mathbf{R}^n$ is denoted by $\mathcal{S}(\mathbf{R}^n)$, its topological dual by $\mathcal{S}'(\mathbf{R}^n)$, and the vector-valued counterparts by $\mathcal{S}(\mathbf{R}^n, E)$ and $\mathcal{S}'(\mathbf{R}^n, E)$, respectively. The symbol $\hookrightarrow$ is used to indicate set-theoretical embeddings which are in addition continuous, e.g. the embeddings

$$\mathcal{S}(\mathbf{R}^n, E) \hookrightarrow L_p(\mathbf{R}^n, E) \hookrightarrow \mathcal{S}'(\mathbf{R}^n, E)$$

are continuous if $1 \le p \le \infty$.

By $C_{ub}(\mathbf{R}^n, E)$ we denote the set of E-valued bounded, uniformly continuous functions. The space $C^m_{ub}(\mathbf{R}^n, E)$ is the set of all E-valued functions which belong together with all of their derivatives of order $\le m$ to $C_{ub}(\mathbf{R}^n, E)$. All theses paces are equipped with their standard norms.

Finally, we mention two inequalities which are well-known in the scalar situation and rather helpful in the vector-valued case, too.

Lemma 2.1 *(Bernstein-Nikol'skij inequality). Let $0 < p_1 \le p_2 \le \infty$ and let $f \in L_{p_1}(\mathbf{R}^n, E) \cap \mathcal{S}'(\mathbf{R}^n, E)$ such that*

$$\operatorname{supp} \mathcal{F} f \subset \{\xi : \quad |\xi| \le b\} .$$

Then for all multi-indices α there exists a positive constant $C = C(p_1, p_2, \alpha)$ such that

$$\| D^\alpha f \,|L_{p_2}(\mathbf{R}^n, E)\| \le C \cdot b^{|\alpha| + n\left(\frac{1}{p_1} - \frac{1}{p_2}\right)} \, \| f \,|L_{p_1}(\mathbf{R}^n, E)\| \tag{1}$$

holds.

Remark 2.2 For this range of parameters a proof of (1) in the vector-valued setting is indicated in [55, 15.3] (with a reference to [53, 1.3.2]). For a complete proof we refer to [49] where we followed [53, 1.3.2], but see also [48] for details.

Let $(M\|f\|_E)$ denote the Hardy-Littlewood maximal function of $\| f(x) \|_E$. The scalar case of the following inequality is treated in Triebel [53, 1.3.1, 1.6.2] and its proof carries over nearly verbatim.

Lemma 2.3 *Let $0 < r < \infty$, $f \in L_\infty(\mathbf{R}^n, E)$, and $\operatorname{supp} \mathcal{F} f \subset [-2^j, 2^j]^n$. Then there exist constants C_1, C_2 independent of f and j such that*

$$\sup_{y \in \mathbf{R}^n} \frac{\|\nabla f(x-y)\|_E}{(1 + 2^j |y|)^{\frac{n}{r}}} \le C_1 \cdot 2^j \sup_{y \in \mathbf{R}^n} \frac{\|f(x-y)\|_E}{(1 + 2^j |y|)^{\frac{n}{r}}} \le C_2 \cdot 2^j \left(M\|f\|^r_E(x)\right)^{\frac{1}{r}} .$$

The scalar version of the next inequality is due to Stein [52, 2.2, pp. 56/57] and its proof carries over verbatim. For two spaces A, B the symbol $A + B$ has the usual meaning.

Lemma 2.4 *Assume that $\mathcal{F}^{-1}\varphi$, $\varphi \in \mathcal{S}'(\mathbf{R}^n)$, has a radial majorant m that is non-increasing and integrable. Then there exists a constant C (independent of φ) such that*

$$\sup_{\lambda > 0} \Big\| \mathcal{F}^{-1} \Big[\varphi(\lambda\xi) \, \mathcal{F} f(\xi)\Big](x) \Big\|_E \le C \left(\int_{\mathbf{R}^n} |m(y)| \, dy \right) \Big(M\|f\|_E \Big)(x)$$

holds for all $x \in \mathbf{R}^n$ and all regular E-valued distributions $f \in L_1(\mathbf{R}^n, E) + L_\infty(\mathbf{R}^n, E)$.

2.2 The Fourier-analytic definition

For us it is most convenient to introduce Besov and Lizorkin-Triebel spaces by means of the so-called decomposition method. It has the disadvantage of some non-transparency but it is most close to Fourier analysis.

Let ψ be an infinitely differentiable function such that $0 \leq \psi(x) \leq 1$, $\psi(x) = 1$ if $|x| \leq A$, and $\psi(x) = 0$ if $|x| > B$ for some numbers $0 < A < B < \infty$. Then we put

$$\varphi_0(x) = \psi(x)\,, \quad \varphi_1(x) = \varphi_0(x/2) - \varphi_0(x)\,, \quad \varphi_j(x) = \varphi_1(2^{-j+1}x)\,,$$

$j = 2, 3, \ldots$. Obviously

$$\sum_{j=0}^{\infty} \varphi_j(x) = 1\,, \qquad x \in \mathbf{R}^n\,.$$

We denote the set of all such smooth dyadic decompositions of unity by $\Phi_{A,B}(\mathbf{R}^n)$.

For any $f \in \mathcal{S}'(\mathbf{R}^n, E)$ it holds

$$f = \sum_{j=0}^{\infty} \mathcal{F}^{-1}[\varphi_j(\xi)\, \mathcal{F} f(\xi)](x)$$

(convergence in $\mathcal{S}'(\mathbf{R}^n, E)$), see e.g. [48] and [3]. In what follows we work quite often with such decompositions. For brevity we shall use the following abbreviation

$$f_j(x) = \mathcal{F}^{-1}[\varphi_j(\xi)\, \mathcal{F} f(\xi)](x), \qquad j = 0, 1, \ldots\,, \quad f \in \mathcal{S}'(\mathbf{R}^n, E).$$

Definition 2.5 Let $\{\varphi_j\}_{j=0}^{\infty} \in \Phi_{A,B}(\mathbf{R}^n)$. Let $-\infty < s < \infty$ and $1 \leq q \leq \infty$.
(i) Let $1 \leq p \leq \infty$. The Besov space $B^s_{p,q}(\mathbf{R}^n, E)$ is the set of all $f \in \mathcal{S}'(\mathbf{R}^n, E)$ such that

$$\| f \,|B^s_{p,q}(\mathbf{R}^n, E)\,\| = \left(\sum_{j=0}^{\infty} 2^{jsq}\, \| f_j \,|L_p(\mathbf{R}^n, E)\|^q \right)^{1/q} < \infty\,,$$

with the usual modification if $q = \infty$.
(ii) Let $1 \leq p < \infty$. The Lizorkin-Triebel space $F^s_{p,q}(\mathbf{R}^n, E)$ is the set of all $f \in \mathcal{S}'(\mathbf{R}^n, E)$ such that

$$\| f \,|F^s_{p,q}(\mathbf{R}^n, E)\,\| = \left\| \left(\sum_{j=0}^{\infty} 2^{jsq}\, \| f_j(\cdot) \,\|^q_E \right)^{1/q} \Big| L_p(\mathbf{R}^n) \right\| < \infty\,,$$

with the usual modification if $q = \infty$.

Remark 2.6 In the scalar case the above definitions go back to papers by Peetre, Nikol'skij, Lizorkin, and Triebel. For vector-valued Besov spaces we refer to [48], [1, 2, 3, 4], [6], [18, 19, 20], [40] and [51]. There exist further and older approaches to vector-valued Besov spaces, for example via interpolation, moduli of continuity, or simply by differences, which lead to the above spaces (in the sense of equivalent norms) in certain cases. We refer to Grisvard [24], Muramatu [35], König [31], and Pietsch [41].
The spaces $F^s_{p,q}(\mathbf{R}^n, E)$ have been introduced in [55, Chapter 15].

Proposition 2.7 *The spaces defined above are Banach spaces. They are independent of the chosen system* $\{\varphi_j\}_{j=0}^{\infty} \in \Phi_{A,B}(\mathbf{R}^n)$ *and also of* A *and* B *in the sense of equivalent norms. We have the chain of continuous embeddings*

$$\mathcal{S}(\mathbf{R}^n, E) \hookrightarrow B^s_{p,q}(\mathbf{R}^n, E)\ ,\ F^s_{p,q}(\mathbf{R}^n, E) \hookrightarrow \mathcal{S}'(\mathbf{R}^n, E)\,.$$

Remark 2.8 Proposition 2.7 is a consequence of a Fourier multiplier assertion for entire analytic E-valued functions, cf. Triebel [55, 15.4]. Based on this tool the proof in the vector-valued situation is the same as in the scalar case, cf. e.g. [53, 2.3.2]. For the embeddings see [53, 2.3.3] and [48].

Remark 2.9 We do not distinguish between spaces equipped with equivalent norms.

Definition 2.10 (i) Let $1 \le p \le \infty$ and let m be a natural number. Then we define

$$W_p^m(\mathbf{R}^n, E) = \Big\{ f \in L_p(\mathbf{R}^n, E) : \ D^\alpha f \in L_p(\mathbf{R}^n, E) \quad \text{for all} \quad |\alpha| \le m \Big\}$$

and put

$$\| \, f \, | W_p^m(\mathbf{R}^n, E) \, \| = \sum_{|\alpha| \le m} \| \, D^\alpha f \, | L_p(\mathbf{R}^n, E) \| \, .$$

(ii) Let $1 < p < \infty$ and let $s \in \mathbf{R}$. Then we define

$$H_p^s(\mathbf{R}^n, E) = \Big\{ f \in \mathcal{S}'(\mathbf{R}^n, E) : \ \mathcal{F}^{-1}[(1 + |\xi|^2)^{s/2} \, \mathcal{F} f(\xi)](x) \in L_p(\mathbf{R}^n, E) \Big\} \tag{2}$$

and put

$$\| \, f \, | H_p^s(\mathbf{R}^n, E) \, \| = \| \, \mathcal{F}^{-1}[(1 + |\xi|^2)^{s/2} \, \mathcal{F} f(\xi)](x) \, | L_p(\mathbf{R}^n, E) \| \, .$$

Remark 2.11 (UMD spaces). It is well-known (cf. McConnell [34], Zimmermann [60]), that

$$H_p^m(\mathbf{R}^n, E) = W_p^m(\mathbf{R}^n, E) \, , \qquad 1 < p < \infty \, ,$$

if and only if the Banach space E has the UMD property (unconditionality of martingale differences) which is equivalent to the property that the Hilbert transform is bounded in $L_2(\mathbf{R}^n, E)$. A survey of characterizations and properties of UMD spaces can be found in Amann [1, Chapter 4.4]and in Pietsch and Wenzel [42]. In particular, UMD spaces are reflexive. As a consequence, a Fourier-analytic approach to vector-valued Sobolev spaces in the sense of (2) is not available for many Banach spaces, which are of interest in operator theory. Let us mention $E = \mathcal{L}(H)$ (where H stands for an arbitrary Hilbert space), the Hölder classes C^s or ℓ_1, L_1, $B^s_{1,q}$, and $F^s_{1,q}$. Nevertheless, the spaces $W_p^m(\mathbf{R}^n, E)$ and $H_p^\sigma(\mathbf{R}^n, E)$ are closely related to the spaces $F^s_{p,q}(\mathbf{R}^n, E)$ and $B^s_{p,q}(\mathbf{R}^n, E)$ via embeddings independent of the UMD property. This paves the way to prove properties of vector-valued Sobolev spaces as we shall see later on.

Remark 2.12 (Littlewood-Paley theory). In the scalar case we have (equivalent norms)

$$H_p^s(\mathbf{R}^n) = F^s_{p,2}(\mathbf{R}^n) \, , \qquad 1 < p < \infty \, . \tag{3}$$

This is based on the Littlewood-Paley-Theorem

$$\| \, f \, | L_p(\mathbf{R}^n) \| \quad \sim \quad \Big\| \Big(\sum_{j=0}^\infty | \, \mathcal{F}^{-1}[\varphi_j \, \mathcal{F} f](x) \, |^2 \Big)^{1/2} \Big| L_p(\mathbf{R}^n) \Big\| , \quad 1 < p < \infty \, ,$$

and the lift property of the spaces $H_p^s(\mathbf{R}^n)$ and $F^s_{p,2}(\mathbf{R}^n)$, see e.g. Lizorkin [33] and Triebel [53]. The arguments in Rubio de Francia and Torrea [45, p. 283] (due to Pisier [43]) show in the E-valued case

$$\| \, f \, | L_p(\mathbf{R}^n, E) \| \sim \Big\| \Big(\sum_{j=0}^\infty \| \, \mathcal{F}^{-1}[\varphi_j \, \mathcal{F} f](x) \, \|_E^2 \Big)^{1/2} \Big| L_p(\mathbf{R}^n) \Big\| , \quad 1 < p < \infty \, ,$$

(or equivalently $L_p(\mathbf{R}^n, E) = F^0_{p,2}(\mathbf{R}^n, E)$) if and only if E can be renormed as a Hilbert space. This extends to the E-valued version of (3) by the lift property (see Propositions 2.16, 2.17 below). Analogously, it holds $F^m_{p,2}(\mathbf{R}^n, E) = W_p^m(\mathbf{R}^n, E) \, , 1 < p < \infty$, if and only if E is Hilbertian.

2.3 Basic properties

Here we summarize certain properties of these spaces which can be proved similarly as in the scalar case.

Proposition 2.13 *Suppose* $s \in \mathbf{R}$, $1 \le p < \infty$, $1 \le q \le \infty$, *and* $1 \le q_1 \le q_2 \le \infty$.
(i) *Then*

$$B^s_{p,q_1}(\mathbf{R}^n, E) \hookrightarrow B^s_{p,q_2}(\mathbf{R}^n, E) \qquad \text{and} \qquad F^s_{p,q_1}(\mathbf{R}^n, E) \hookrightarrow F^s_{p,q_2}(\mathbf{R}^n, E)$$

hold. Further we have

$$B^s_{p,\min(p,q)}(\mathbf{R}^n, E) \hookrightarrow F^s_{p,q}(\mathbf{R}^n, E) \hookrightarrow B^s_{p,\max(p,q)}(\mathbf{R}^n, E)\,.$$

(ii) *If* $s > 0$, *then*

$$B^s_{p,q}(\mathbf{R}^n, E) \hookrightarrow L_p(\mathbf{R}^n, E) \text{ and } F^s_{p,q}(\mathbf{R}^n, E) \hookrightarrow L_p(\mathbf{R}^n, E)\,.$$

(iii) *Suppose* $E_1 \hookrightarrow E_2$. *Then*

$$B^s_{p,q}(\mathbf{R}^n, E_1) \hookrightarrow B^s_{p,q}(\mathbf{R}^n, E_2) \quad \text{and} \quad F^s_{p,q}(\mathbf{R}^n, E_1) \hookrightarrow F^s_{p,q}(\mathbf{R}^n, E_2)\,.$$

The Nikol'skij inequality implies the following embeddings.

Proposition 2.14 *Suppose* $s_0,\ s_1 \in \mathbf{R}^n$, $1 \le p_0 \le p_1 \le \infty$, *and* $1 \le q_0 \le q_1 \le \infty$. *Further, let*

$$s_0 - \frac{n}{p_0} \ge s_1 - \frac{n}{p_1}\,.$$

Then we have the continuous embedding $B^{s_0}_{p_0,q_0}(\mathbf{R}^n, E) \hookrightarrow B^{s_1}_{p_1,q_1}(\mathbf{R}^n, E)$.

Remark 2.15 The scalar case can be found in Nikol'skij [37], see also [53].

Next we describe the lifting property. For a real number σ and $f \in \mathcal{S}'(\mathbf{R}^n, E)$ we put

$$I_\sigma f = \mathcal{F}^{-1}[(1 + |\xi|^2)^{\sigma/2}\, \mathcal{F}f(\xi)]\,.$$

As in the scalar case these mappings are one-to-one on $\mathcal{S}'(\mathbf{R}^n, E)$.

Proposition 2.16 *Let* $-\infty < s, \sigma < \infty$, *and* $1 \le q \le \infty$.
(i) *Let* $1 \le p \le \infty$. *Then* I_σ *maps* $B^s_{p,q}(\mathbf{R}^n, E)$ *isomorphically onto* $B^{s-\sigma}_{p,q}(\mathbf{R}^n, E)$ *and* $\| I_\sigma f\, |B^{s-\sigma}_{p,q}(\mathbf{R}^n, E)\|$ *is an equivalent norm on* $B^s_{p,q}(\mathbf{R}^n, E)$.
(ii) *Let* $1 \le p < \infty$. *Then* I_σ *maps* $F^s_{p,q}(\mathbf{R}^n, E)$ *isomorphically onto* $F^{s-\sigma}_{p,q}(\mathbf{R}^n, E)$ *and* $\| I_\sigma f\, |F^{s-\sigma}_{p,q}(\mathbf{R}^n, E)\|$ *is an equivalent norm on* $F^s_{p,q}(\mathbf{R}^n, E)$.

Proof. We restrict ourselves to a comment. Looking at the scalar case in [53, 2.3.8] it turns out that the proof reduces to Fourier multipliers of (weak) Michlin-Hörmander type. For vector-valued Besov spaces and Lizorkin-Triebel spaces the validity of such Fourier multiplier assertions has been shown in [55, 15.6]. □

In the same way one can adopt the proof of the following supplement, cf. [53, 2.3.8].

Proposition 2.17 *Let* $-\infty < s < \infty$, $m \in \mathbf{N}$, *and* $1 \le q \le \infty$.
(i) *Let* $1 \le p \le \infty$. *Then*

$$\sum_{|\alpha| \le m} \| D^\alpha f\, |B^{s-m}_{p,q}(\mathbf{R}^n, E)\|$$

defines an equivalent norm on $B^s_{p,q}(\mathbf{R}^n, E)$.
(ii) *Let* $1 \le p < \infty$. *Then*

$$\sum_{|\alpha| \le m} \| D^\alpha f\, |F^{s-m}_{p,q}(\mathbf{R}^n, E)\|$$

defines an equivalent norm on $F^s_{p,q}(\mathbf{R}^n, E)$.

Finally we deal with the Fatou property. Let A be continuously embedded into $\mathcal{S}'(\mathbf{R}^n, E)$. Then we say that A has the Fatou property if the following conclusion is true: If $\{g_j\}_j$ is a convergent sequence in $\mathcal{S}'(\mathbf{R}^n, E)$ with limit g and if

$$\sup_{j=0,1,\ldots} \| \, g_j \, |A\| \le C < \infty \, ,$$

then $g \in A$ and $\| \, g \, |A\| \le c \, C$ for some c independent of C.

Proposition 2.18 *All Besov and Lizorkin-Triebel spaces* $B^s_{p,q}(\mathbf{R}^n, E)$ *and* $F^s_{p,q}(\mathbf{R}^n, E)$ *have the Fatou property.*

Remark 2.19 In this generality (but for the scalar case) Franke [15] was the first who noticed this property of Besov and Lizorkin-Triebel spaces (and his proof carries over to the situation treated here). Independently, but restricted to Besov spaces, also Bourdaud and Meyer [7] have used the Fatou property.

2.4 The sandwich argument

To carry over properties of Lizorkin-Triebel spaces to Sobolev spaces we shall use the following embedding.

Theorem 2.20 (i) *Suppose* $1 < p < \infty$ *and* $m \in \mathbf{N}_0$. *Then we have the chain of continuous embeddings*

$$B^m_{p,1}(\mathbf{R}^n, E) \hookrightarrow F^m_{p,1}(\mathbf{R}^n, E) \hookrightarrow W^m_p(\mathbf{R}^n, E) \hookrightarrow F^m_{p,\infty}(\mathbf{R}^n, E) \hookrightarrow B^m_{p,\infty}(\mathbf{R}^n, E)$$

(ii) *Suppose* $1 < p < \infty$ *and* $s \in \mathbf{R}$. *Then we have the continuous embeddings*

$$F^s_{p,1}(\mathbf{R}^n, E) \hookrightarrow H^s_p(\mathbf{R}^n, E) \hookrightarrow F^s_{p,\infty}(\mathbf{R}^n, E) \, .$$

(iii) *Suppose* $m \in \mathbf{N}_0$. *Then we have the chain of continuous embeddings*

$$B^m_{\infty,1}(\mathbf{R}^n, E) \hookrightarrow C^m_{ub}(\mathbf{R}^n, E) \hookrightarrow W^m_\infty(\mathbf{R}^n, E) \hookrightarrow B^m_{\infty,\infty}(\mathbf{R}^n, E) \, .$$

Remark 2.21 A proof can be found in [50].

3 Atomic and subatomic representations of vector-valued Function Spaces

It is our aim to decompose each element of a function space $B^s_{p,q}(\mathbf{R}^n, E)$ resp. $F^s_{p,q}(\mathbf{R}^n, E)$ as a preferably easy (infinite) linear combination of suitable functions. We will present the necessary steps and assertions, mainly without proofs.

In the following we assume $s > 0$, $1 \le p \le \infty$ (resp. $< \infty$), and $1 \le q \le \infty$, which is the range of parameters of interest for the next section. A more detailed treatment can be found in [47], which is based on the remarks in [55, Chapter 15], in particular Theorems 15.8 and 15.11.

3.1 Atomic decompositions

At first we describe the concept of atoms as it was introduced by Frazier and Jawerth [16, 17], see also Triebel [55, 13.3]. Here $Q_{\nu,m} := \{x \in \mathbf{R}^n : |x_i - 2^{-\nu} m_i| \le 2^{-\nu-1}\}$ stands for the cube with sides parallel to the axes, with centre at $2^{-\nu}m$, and side length $2^{-\nu}$ for $m \in \mathbf{Z}^n$ and $\nu \in \mathbf{N}_0$.
Furthermore, with $\lfloor x \rfloor$ we denote the largest integer smaller or equal to $x \in \mathbf{R}$.

Definition 3.1 Let $s > 0$, $1 \le p \le \infty$, $K \in \mathbf{N}_0$, and $d > 1$. A K times differentiable (in the case $K = 0$ continuous) function $a : \mathbf{R}^n \to E$ is called (E-valued) $(s,p)_K$-atom if there exists a number $\nu \in \mathbf{N}_0$ such that

$$\begin{aligned} & supp\; a \subset d \cdot Q_{\nu,m} \text{ for some } m \in \mathbf{Z}, \\ & \|D^\alpha a(x)\|_E \le 2^{-\nu\left(s - \frac{n}{p}\right) + |\alpha|\nu} \text{ for all } |\alpha| \le K. \end{aligned} \qquad (4)$$

In particular, $a_{\nu,m}(\cdot) e_{\nu,m}$ is a vector-valued $(s,p)_K$-atom if $a_{\nu,m}$ is a scalar (i.e. $\mathbf{C}$-valued) $(s,p)_K$-atom and $e_{\nu,m} \in U_E = \{e \in E : \|e\|_E = 1\}$ is fixed.

Furthermore, we introduce the sequence spaces $b_{p,q}$ and $f_{p,q}$, whose use will become clear in the following. See also [55, Definition 13.5].

Definition 3.2 Let $1 \le p \le \infty$, $1 \le q \le \infty$ and

$$\lambda = \{\lambda_{\nu,m} \in \mathbf{C} : \nu \in \mathbf{N}_0, m \in \mathbf{Z}^n\}.$$

We set

$$b_{p,q} := \left\{ \lambda : \|\lambda|b_{p,q}\| = \left(\sum_{\nu=0}^{\infty} \left(\sum_{m \in \mathbf{Z}^n} |\lambda_{\nu,m}|^p \right)^{\frac{q}{p}} \right)^{\frac{1}{q}} < \infty \right\}$$

and

$$f_{p,q} := \left\{ \lambda : \|\lambda|f_{p,q}\| = \left\| \left(\sum_{\nu=0}^{\infty} \sum_{m \in \mathbf{Z}^n} |\lambda_{\nu,m} \chi_{\nu,m}^{(p)}(\cdot)|^q \right)^{\frac{1}{q}} \Big| L_p \right\| < \infty \right\}$$

(modified in the cases $p = \infty$ or $q = \infty$), where $\chi_{\nu,m}^{(p)}$ is the L_p-normalized characteristic function of the cube $Q_{\nu,m}$.

The scalar version of the following lemma can be found in [55, Corollary 13.9]. It is the starting point of our representation.

Lemma 3.3 *Let $1 \le p \le \infty$ (resp. $< \infty$), $1 \le q \le \infty$, and $s > 0$. Let $K \in \mathbf{N}_0$ with $K \ge 1 + \lfloor s \rfloor$. Then*

$$\sum_{\nu=0}^{\infty} \sum_{m \in \mathbf{Z}^n} \lambda_{\nu,m} a_{\nu,m}(x)$$

converges unconditionally in $\mathcal{S}'(\mathbf{R}^n, E)$, where $a_{\nu,m}$ are E-valued $(s,p)_K$-atoms centred at $2^{-\nu}m$ ($\nu \in \mathbf{N}_0, m \in \mathbf{Z}$) and $\lambda \in b_{p,q}$ or $\lambda \in f_{p,q}$.

Next we give an equivalent characterization of the above defined Besov and Lizorkin-Triebel spaces by local means which has been proven in [47, Chapter 3]. The scalar version goes back to [46] and to the more general, but more technical approach in [54, Chapter 2].

Let $k_0, k^0 \in \mathcal{S}(\mathbf{R}^n)$, $\hat{k_0}(0) \neq 0$, $\hat{k^0}(0) \neq 0$ such that *supp* k_0, *supp* $k^0 \subset B = \{x \in \mathbf{R}^n : |x| \le 1\}$ and $k_j^N := 2^{jn} \left(\Delta^N k^0\right)(2^j \cdot)$.

Proposition 3.4 *Let $2N > s > 0$.*

(i) Let $1 \le p \le \infty$ and $1 \le q \le \infty$. Then

$$\|f|B_{p,q}^s(\mathbf{R}^n, E)\|_{k_0,k^N} := \|k_0 * f|L_p(E)\| + \left(\sum_{j=1}^{\infty} 2^{jsq} \|k_j^N * f|L_p(E)\|^q \right)^{\frac{1}{q}}$$

(modified for $q = \infty$) is an equivalent norm for $\|\cdot|B_{p,q}^s(\mathbf{R}^n, E)\|$. It holds

$$B_{p,q}^s(\mathbf{R}^n, E) = \left\{ f \in \mathcal{S}'(\mathbf{R}^n, E) : \|f|B_{p,q}^s(\mathbf{R}^n, E)\|_{k_0,k^N} < \infty \right\}.$$

(ii) Let $1 \le p < \infty$ and $1 \le q \le \infty$. Then

$$\|f|F_{p,q}^s(\mathbf{R}^n, E)\|_{k_0,k^N} := \|k_0 * f|L_p(E)\| + \left\| \left(\sum_{j=1}^{\infty} 2^{jsq} \|k_j^N * f|E\|^q \right)^{\frac{1}{q}} \Big| L_p \right\|$$

(modified for $q = \infty$) is an equivalent norm for $\|\cdot|F_{p,q}^s(\mathbf{R}^n, E)\|$. It holds

$$F_{p,q}^s(\mathbf{R}^n, E) = \left\{ f \in \mathcal{S}'(\mathbf{R}^n, E) : \|f|F_{p,q}^s(\mathbf{R}^n, E)\|_{k_0,k^N} < \infty \right\}.$$

Now we are ready to give the first part of the theorem on atomic decompositions.

Proposition 3.5 *Let $K \in \mathbf{N}_0$ with $K \geq 1 + \lfloor s \rfloor$.*
(i) Let $1 \leq p \leq \infty$, $1 \leq q \leq \infty$, and $s > 0$. Let $\lambda \in b_{p,q}$ and let $a_{\nu,m}$ be E-valued $(s,p)_K$-atoms centred at $2^{-\nu}m$ ($\nu \in \mathbf{N}_0$, $m \in \mathbf{Z}$). Then each $f \in \mathcal{S}'(\mathbf{R}^n, E)$ which can be represented by

$$f = \sum_{\nu \in \mathbf{N}_0} \sum_{m \in \mathbf{Z}^n} \lambda_{\nu,m} a_{\nu,m}$$

in $\mathcal{S}'(\mathbf{R}^n, E)$ belongs to $B^s_{p,q}(\mathbf{R}^n, E)$. Furthermore, there exists a constant c independent of f, λ, and $(a_{\nu,m})_{\nu,m}$ such that

$$\|f|B^s_{p,q}(\mathbf{R}^n, E)\| \leq c\|\lambda|b_{p,q}\|.$$

(ii) Let $1 \leq p < \infty$, $1 \leq q \leq \infty$, and $s > 0$. Let $\lambda \in f_{p,q}$ and let $a_{\nu,m}$ be E-valued $(s,p)_K$-atoms centred at $2^{-\nu}m$ ($\nu \in \mathbf{N}_0$, $m \in \mathbf{Z}$). Then each $f \in \mathcal{S}'(\mathbf{R}^n, E)$ which can be represented by

$$f = \sum_{\nu \in \mathbf{N}_0} \sum_{m \in \mathbf{Z}^n} \lambda_{\nu,m} a_{\nu,m}$$

in $\mathcal{S}'(\mathbf{R}^n, E)$ belongs to $F^s_{p,q}(\mathbf{R}^n, E)$. Furthermore, there exists a constant c independent of f, λ, and $(a_{\nu,m})_{\nu,m}$ such that

$$\|f|F^s_{p,q}(\mathbf{R}^n, E)\| \leq c\|\lambda|f_{p,q}\|.$$

Proof. One relies on the equivalent quasi-norm from Proposition 3.4 as in the scalar case (see [55, Theorem 13.8]). A detailed proof can be found in [47, Proposition 4.4]. □

Now the question will be whether all elements of the function space can be represented in such a way. A positive answer in the scalar (i.e. $E = \mathbf{C}$) case has been given for instance in [55, Theorem 13.8]. For the vector-valued case we will slightly modify the way as described in [55, Theorem 15.8]. To explain this in more detail (as in [55, 12.2] in the scalar case) we take an $f \in \mathcal{S}(\mathbf{R}^n, E)$ and define the functions $u(x,t)$ for $x \in \mathbf{R}^n$, $t > 0$ by

$$u(x,t) := (e^{-t|\cdot|}\hat{f})\check{}(x) = d_n \left(f * \frac{t}{(|\cdot|^2 + t^2)^{\frac{n+1}{2}}} \right)(x).$$

It follows

$$u(x,t) \to f(x) \text{ for } t \to 0 \text{ and } t^k \frac{\partial^k u(x,t)}{\partial t^k} \to 0 \text{ for } t \to 0 \text{ and for all } k \in \mathbf{N}$$

uniformly in x. Now we obtain by iterated partial integration and with suitable constants d^k_l for $k \in \mathbf{N}$ and $l \in \{0, \ldots, k-1\}$

$$\begin{aligned}
\int_a^b t^{k-1} \frac{\partial^k u(x,t)}{\partial t^k}\, dt &= \tau^{k-1} \frac{\partial^{k-1} u(x,\tau)}{\partial \tau^{k-1}}\Big|_a^b - (k-1) \int_a^b t^{k-2} \frac{\partial^{k-1} u(x,t)}{\partial t^{k-1}} \\
&= \ldots \\
&= \sum_{l=0}^{k-1} d^k_l b^l \frac{\partial^l u(x,b)}{\partial t^l} - \sum_{l=0}^{k-1} d^k_l a^l \frac{\partial^l u(x,a)}{\partial t^l}.
\end{aligned}$$

Therefrom we get the equality

$$\sum_{l=0}^{k-1} d^k_l \tau^l \frac{\partial^l u(x,\tau)}{\partial t^l}\Big|_0^1 = \sum_{\nu=0}^{\infty} \int_{2^{-\nu-1}}^{2^{-\nu}} t^{k-1} \frac{\partial^k u(x,t)}{\partial t^k}\, dt.$$

By our above considerations on limits it follows

$$f(x)=c\sum_{\nu=0}^{\infty}\int_{2^{-\nu-1}}^{2^{-\nu}} t^{k-1}\frac{\partial^k u(x,t)}{\partial t^k}\,dt+\sum_{l=0}^{k-1}c_l^k\frac{\partial^l u(x,1)}{\partial t^l} \tag{5}$$

with appropriate constants c_l^k with $k\in\mathbf{N}$ and $l\in\{0,\dots,k-1\}$. The right-hand side of (5) is called a harmonic representation of f.

In the following we adapt [55, Theorem 12.5], where the scalar case is treated. The functions $u(x,t)$ and their partial derivatives make sense for $f\in B^s_{p,q}(\mathbf{R}^n,E)$ or $f\in F^s_{p,q}(\mathbf{R}^n,E)$ and it holds

Lemma 3.6 *Let $s>0$, $1\le p\le\infty$ (resp. $<\infty$), and $1\le q\le\infty$. If one chooses $k\in\mathbf{N}$ large enough, then the right-hand side of* (5) *converges in $\mathcal{S}'(\mathbf{R}^n,E)$ to f for $f\in B^s_{p,q}(\mathbf{R}^n,E)$ resp. $F^s_{p,q}(\mathbf{R}^n,E)$.*

Now we derive a representation of the elements of the function spaces $B^s_{p,q}(\mathbf{R}^n,E)$ resp. $F^s_{p,q}(\mathbf{R}^n,E)$ which leads us to the inverse of Proposition 3.5. Hereby we adapt [55, 13.10].

Let $f\in B^s_{p,q}(\mathbf{R}^n,E)$ resp. $F^s_{p,q}(\mathbf{R}^n,E)$. If we choose k sufficiently large, then

$$f(x)=c\sum_{\nu=0}^{\infty}\int_{2^{-\nu-1}}^{2^{-\nu}} t^{k}\frac{\partial^k u(x,t)}{\partial t^k}\,\frac{dt}{t}+\sum_{l=0}^{k-1}c_l^k\frac{\partial^l u(x,1)}{\partial t^l}$$

with convergence in $\mathcal{S}'(\mathbf{R}^n,E)$ by Lemma 3.6.

Let $\mu\in\mathbf{N}$ be fixed. Let $\nu\in\mathbf{N}$, $\nu\ge\mu$, $m\in\mathbf{Z}^n$ and $l\in\{0,\cdots 2^\mu-1\}$. By $B_{\nu,m,l}$ we denote the cubes in $\mathbf{R}^{n+1}_+ := \{(x,t)\in\mathbf{R}^{n+1}, t>0\}$ with centre $(2^{-\nu}m, 2^{-\nu+\mu}+l2^{-\nu})$ and radius $2^{-\nu+\mu-2}$. We decompose the rectangles $Q_{\nu,m}\times(2^{-\nu+\mu},2^{-\nu+\mu+1})$ in 2^μ cubes of side length $2^{-\nu}$. Now we define

$$\lambda_{\nu,m} := 2^{\nu\left(s-\frac{n}{p}\right)}2^{-\nu k}\sup\|\frac{\partial^k u(y,t)}{\partial t^k}\|_E \text{ for } \nu>\mu, m\in\mathbf{Z}^n, \tag{6}$$

where we take the supremum over the set

$$\left\{(y,t)\in\mathbf{R}^{n+1} : |2^{-\nu}m-y|\le d2^{-\nu+\mu-1}, d^{-1}2^{-\nu+\mu}\le t\le d2^{-\nu+\mu+1}\right\},$$

for a $d>0$ which will be chosen sufficiently large later on.

In the case $\mu=\nu$ we put

$$\lambda_{\mu,m} := \sum_{l=0}^{k-1}|c_l^k|\sup\|\frac{\partial^l u(y,t)}{\partial t^l}\|_E \text{ for } m\in\mathbf{Z}^n,$$

where we take the supremum over the set

$$\left\{(y,t)\in\mathbf{R}^{n+1} : |2^{-\nu}m-y|\le d, \frac{1}{2d}\le t\le\frac{3}{2}d\right\}.$$

We obtain

$$\|\lambda|f_{p,q}\|\le c2^{\mu\delta}\|f|F^s_{p,q}(\mathbf{R}^n,E)\| \tag{7}$$

for a suitable $\delta>0$ and for a c independent of μ and an analogous result for $B^s_{p,q}(\mathbf{R}^n,E)$. For a proof of this estimate see [47, p. 36]. Now we choose $\psi\in\mathcal{S}(\mathbf{R}^n)$ with compact support and

$$\sum_{m\in\mathbf{Z}^n}\psi(x-m)=1 \text{ for all } x\in\mathbf{R}^n. \tag{8}$$

Let c be the constant from (5). If $\nu>\mu$ and $m\in\mathbf{Z}^n$, we put

$$a_{\nu,m}(x) := \sum_{l=0}^{2^\mu-1}a_{\nu,m,l}(x) \text{ with}$$

$$a_{\nu,m,l}(x) := c\lambda_{\nu,m}^{-1}\psi(2^\nu x-m)\int_{2^{-\nu+\mu}+l2^{-\nu}}^{2^{-\nu+\mu}+(l+1)2^{-\nu}} t^k\frac{\partial^k u(x,t)}{\partial t^k}\,\frac{dt}{t}$$

and in the case $\nu = \mu$ we define for $m \in \mathbf{Z}^n$

$$a_{\mu,m}(x) := \sum_{l=0}^{k-1} a_{\mu,m,l}(x) \text{ with } a_{\mu,m,l}(x) := c_l^k \lambda_{\mu,m}^{-1} \psi(2^\mu x - m) \frac{\partial^l u(x,1)}{\partial t^l}.$$

Then we obtain (convergence in $\mathcal{S}'(\mathbf{R}^n, E)$)

$$f = \sum_{\nu=\mu}^{\infty} \sum_{m \in \mathbf{Z}^n} \lambda_{\nu,m} a_{\nu,m}.$$

This is the desired representation. One can show that the $a_{\nu,m}$ behave like E-valued $(s,p)_K$-atoms for all $K \in \mathbf{N}$. We call these atoms and the found representation for f "harmonic".

Theorem 3.7 *Let $K \in \mathbf{N}_0$ with $K \geq 1 + \lfloor s \rfloor$.*
(i) Let $1 \leq p \leq \infty$, $1 \leq q \leq \infty$, $s > 0$. Then $f \in \mathcal{S}'(\mathbf{R}^n, E)$ belongs to $B^s_{p,q}(\mathbf{R}^n, E)$ if and only if it can be represented by

$$f = \sum_{\nu \in \mathbf{N}_0} \sum_{m \in \mathbf{Z}^n} \lambda_{\nu,m} a_{\nu,m}.$$

Here $a_{\nu,m}$ are E-valued $(s,p)_K$-atoms centred at $2^{-\nu}m$ ($\nu \in \mathbf{N}_0, m \in \mathbf{Z}$) and $\lambda \in b_{p,q}$. Furthermore, we have

$$\| f | B^s_{p,q}(\mathbf{R}^n, E) \| \sim \inf \| \lambda | b_{p,q} \|$$

in the sense of equivalence of norms, where the infimum on the right-hand side is taken over all admissible representations for f.
(ii) Let $1 \leq p < \infty$, $1 \leq q \leq \infty$, $s > 0$. Then $f \in \mathcal{S}'(\mathbf{R}^n, E)$ belongs to $F^s_{p,q}(\mathbf{R}^n, E)$ if and only if it can be represented by

$$f = \sum_{\nu \in \mathbf{N}_0} \sum_{m \in \mathbf{Z}^n} \lambda_{\nu,m} a_{\nu,m}.$$

Here $a_{\nu,m}$ are E-valued $(s,p)_K$-atoms centred at $2^{-\nu}m$ ($\nu \in \mathbf{N}_0, m \in \mathbf{Z}$) and $\lambda \in f_{p,q}$. Furthermore, we have

$$\| f | F^s_{p,q}(\mathbf{R}^n, E) \| \sim \inf \| \lambda | f_{p,q} \|$$

in the sense of equivalence of norms, where the infimum on the right-hand side is taken over all admissible representations for f.

P r o o f. The assertions follow from Proposition 3.5 and from the proven representation observing (7) resp. the analogous result for $B^s_{p,q}(\mathbf{R}^n, E)$. One has to show that $a_{\nu,m}$ is an E-valued $(s,p)_K$-atom. Here the coefficients $\lambda_{\nu,m}$ even vanish for $\nu < \mu$.
A more detailed proof can be found in [47, Proposition 4.7]. □

3.2 Subatomic decompositions

The aim of the following section is to obtain a further simplification of the atomic decomposition of $f \in B^s_{p,q}(\mathbf{R}^n, E)$ resp. $F^s_{p,q}(\mathbf{R}^n, E)$. As a starting point we use the harmonic representation from the last section. We follow [55, Chapter 14], where the scalar case is treated.

Definition 3.8 Let $\psi \in \mathcal{S}(\mathbf{R}^n)$ with $supp\ \psi \subset d \cdot Q_{0,0}$ for a $d > 1$ and

$$\sum_{m \in \mathbf{Z}^n} \psi(x - m) = 1.$$

Let $s > 0$, $1 \leq p \leq \infty$, and $\gamma \in \mathbf{N}_0^n$. We put $\psi^\gamma(x) := x^\gamma \psi(x)$. Then we call

$$(\gamma qu)_{\nu,m}(x) = 2^{-\nu\left(s - \frac{n}{p}\right)} \psi^\gamma(2^\nu x - m)$$

a (s,p)-γ-quark for $Q_{\nu,m}$.

Remark 3.9 For the derivatives of the (s,p)-γ-quarks we have

$$|D^\alpha\left(\psi^\gamma(2^\nu x-m)\right)|\le c\,2^{|\alpha|\nu}2^{\varkappa|\gamma|},$$

where c and $\varkappa$ depend on α but not on γ, ν, or m. So the $(\gamma qu)_{\nu,m}(x)$ are $(s,p)_K$-atoms centred at $2^{-\nu}m$ up to a constant.

Now we will simplify the representation of $f\in B^s_{p,q}(\mathbf{R}^n,E)$ resp. $F^s_{p,q}(\mathbf{R}^n,E)$. We prove [55, Theorem 15.8] with the given restrictions for s,p, and q.

Theorem 3.10 *(i) Let $1\le p\le\infty$, $1\le q\le\infty$, and $s>0$. Let $(\gamma qu)_{\nu,m}$ be (s,p)-γ-quarks for a given function $\psi\in\mathcal{S}(\mathbf{R}^n)$ with compact support satisfying property* (8)*. Then there exists a $\varkappa>0$ such that for all $\mu\ge\varkappa$ it is valid that $f\in\mathcal{S}'(\mathbf{R}^n,E)$ belongs to $B^s_{p,q}(\mathbf{R}^n,E)$ if and only if it can be represented as*

$$f=\sum_{\gamma\in\mathbf{N}_0^n}\sum_{\nu=0}^{\infty}\sum_{m\in\mathbf{Z}^n}\varrho^\gamma_{\nu,m}(\gamma qu)_{\nu,m}(\cdot)\,e^\gamma_{\nu,m}$$

in $\mathcal{S}'(\mathbf{R}^n,E)$ with $e^\gamma_{\nu,m}\in U_E$ and

$$\sup_{\gamma\in\mathbf{N}_0}2^{\mu|\gamma|}\|\varrho^\gamma|b_{p,q}\|<\infty.$$

Furthermore, it holds in the sense of equivalence of norms

$$\|f|B^s_{p,q}(\mathbf{R}^n,E)\|\sim\inf\sup_\gamma 2^{\mu|\gamma|}\|\varrho^\gamma|b_{p,q}\|,$$

where the infimum on the right-hand side is taken over all admissible representations of f.
(ii) Let $1\le p\le\infty$, $1\le q\le\infty$, and $s>0$. The quarks have the same meaning as in (i). Then there exists a $\varkappa>0$ such that for all $\mu\ge\varkappa$ it is valid that $f\in\mathcal{S}'(\mathbf{R}^n,E)$ belongs to $F^s_{p,q}(\mathbf{R}^n,E)$ if and only if it can be represented as

$$f=\sum_{\gamma\in\mathbf{N}_0^n}\sum_{\nu=0}^{\infty}\sum_{m\in\mathbf{Z}^n}\varrho^\gamma_{\nu,m}(\gamma qu)_{\nu,m}(\cdot)\,e^\gamma_{\nu,m}\tag{9}$$

in $\mathcal{S}'(\mathbf{R}^n,E)$ with $e^\gamma_{\nu,m}\in U_E$ and

$$\sup_\gamma 2^{\mu|\gamma|}\|\varrho^\gamma|f_{p,q}\|<\infty.\tag{10}$$

Furthermore, it holds in the sense of equivalence of norms

$$\|f|F^s_{p,q}(\mathbf{R}^n,E)\|\sim\inf\sup_{\gamma\in\mathbf{N}_0}2^{\mu|\gamma|}\|\varrho^\gamma|f_{p,q}\|,$$

where the infimum on the right-hand side is taken over all admissible representations of f.

P r o o f. We only consider the case $F^s_{p,q}(\mathbf{R}^n,E)$. The proof for $B^s_{p,q}(\mathbf{R}^n,E)$ can be organized analogously. Let $f\in\mathcal{S}'(\mathbf{R}^n,E)$ be represented by (9) with the condition (10). Then

$$f^\gamma:=\sum_{\nu=0}^{\infty}\sum_{m\in\mathbf{Z}^n}\varrho^\gamma_{\nu,m}(\gamma qu)_{\nu,m}(\cdot)e^\gamma_{\nu,m}$$

is represented as a sum of atoms (up to a constant) by Remark 3.9. Here $(\gamma qu)_{\nu,m}e^\gamma_{\nu,m}$ are E-valued $(s,p)_K$-atoms for every $K\in\mathbf{N}_0$, where one has to keep in mind a normalization constant $c\cdot 2^{\varkappa\gamma}$ with c depending on K but independent of γ. Thus we obtain by Proposition 3.5 that $f^\gamma\in F^s_{p,q}(\mathbf{R}^n,E)$ and that there exists a constant $c''>0$ such that it holds

$$\|f^\gamma|F^s_{p,q}(\mathbf{R}^n,E)\|\le c''2^{\varkappa|\gamma|}\|\varrho^\gamma|f_{p,q}\|.$$

Therefore, if we take $\mu > \varkappa$ for granted, it follows from (10) and a standard Minkowski/Hölder argument that

$$f = \sum_{\gamma \in \mathbf{N}_0^n} f^\gamma \quad \text{with} \quad \| f | F^s_{p,q}(\mathbf{R}^n, E) \| \le C \sup_\gamma 2^{\mu|\gamma|} \| \varrho^\gamma | f_{p,q} \|$$

in $F^s_{p,q}(\mathbf{R}^n, E)$.

Now we show that each $f \in F^s_{p,q}(\mathbf{R}^n, E)$ can be represented in the desired way. From Theorem 3.7 and the previous remarks we obtain the optimal decomposition

$$f = \sum_{\nu \ge \mu} \sum_{m \in \mathbf{Z}^n} \lambda_{\nu,m} a_{\nu,m} \tag{11}$$

with (in case of $\nu > \mu$)

$$\begin{aligned} a_{\nu,m,l}(x) &= c\, \lambda^{-1}_{\nu,m} \psi(2^\nu x - m) \int_{2^{-\nu+\mu}+l2^{-\nu}}^{2^{-\nu+\mu}+(l+1)2^{-\nu}} t^k \frac{\partial^k u(x,t)}{\partial t^k} \, \frac{dt}{t}, \\ a_{\nu,m}(x) &= \sum_{l=0}^{2^\mu-1} a_{\nu,m,l}(x) \quad \text{and} \quad \| \lambda | f_{p,q} \| \le c 2^{\mu\delta} \| f | F^s_{p,q}(\mathbf{R}^n, E) \|. \end{aligned} \tag{12}$$

We expand the infinitely often differentiable functions $\frac{\partial^k u(x,t)}{\partial t^k}$ into a Taylor series, with center $(2^{-\nu}m, 2^{-\nu+\mu} + l2^{-v})$ of the balls $B_{\nu,m,l}$ and integrate over t.

Keeping in mind the definition of $\lambda_{\nu,m}$ in (6) we obtain

$$\begin{aligned} a_{\nu,m,l}(x) &= c\, \lambda^{-1}_{\nu,m} \psi(2^\nu x - m) \int_{2^{-\nu+\mu}+l2^{-\nu}}^{2^{-\nu+\mu}+(l+1)2^{-\nu}} t^k \frac{\partial^k u(x,t)}{\partial t^k} \, \frac{dt}{t} \\ &\equiv \sum_{\gamma \in \mathbf{N}_0^N} \tilde{c}_\gamma \, (2^\nu x - m)^\gamma \psi(2^\nu x - m) \end{aligned}$$

with

$$\begin{aligned} \| \tilde{c}_\gamma \|_E &\le c\, 2^{-\nu|\gamma|} \lambda^{-1}_{\nu,m} \sum_{\beta=0}^\infty \frac{2^{(\nu-\mu+2)(|\gamma|+\beta)}}{\gamma!\beta!} \, c_{\gamma,\beta} \int\limits_{2^{-\nu+\mu}+l2^{-\nu}}^{2^{-\nu+\mu}+(l+1)2^{-\nu}} (t - 2^{-\nu+\mu} - l2^{-\nu})^\beta t^k \, \frac{dt}{t} \\ &\le c'\, 2^{-\nu\left(s-\frac{n}{p}\right)} 2^{\mu k} 2^{(-\mu+2)|\gamma|} \tau^{-|\gamma|} \sum_{\beta=0}^\infty \tau^{-\beta} 2^{(-\mu+2)\beta}. \end{aligned}$$

where $c_{\gamma,\beta}$ are the Taylor coefficients of $\frac{\partial^k u(x,t)}{\partial t^k}$ which can be estimated from above uniformly, see [47, Lemma 4.11] and the following steps, with an appropriate $\tau \in (0,1)$. The sum over β converges if we choose μ large enough and we get

$$a_{\nu,m,l}(x) = \sum_{\gamma \in \mathbf{N}_0^n} \eta^\gamma_{\nu,m,l} (\gamma q u)_{\nu,m}(x) \;\; \text{with} \;\; \| \eta^\gamma_{\nu,m,l} \|_E \le c'' 2^{\mu k} (\tau^{-1} 2^{-\mu+2})^{|\gamma|}.$$

If we replace μ by $M\mu$ afterwards, where $M \in \mathbf{N}$ is sufficiently large, and sum over $l = 0, \ldots, 2^\mu - 1$ in (12), we arrive at

$$a_{\nu,m}(x) = \sum_{\gamma \in \mathbf{N}_0^n} \eta^\gamma_{\nu,m} (\gamma q u)_{\nu,m}(x) \;\; \text{with} \;\; \| \eta^\gamma_{\nu,m} \|_E \le C 2^{\mu\delta} 2^{-\mu|\gamma|} \tag{13}$$

for certain $C > 0$ and $\delta > 0$ which do not depend on μ and γ.

The case $\nu = \mu$ can be treated analogously, even simpler. Hence we obtain from (11)

$$f = \sum_{\gamma \in \mathbf{N}_0^n} \sum_{\nu=0}^\infty \sum_{m \in \mathbf{Z}^n} \lambda^\gamma_{\nu,m} (\gamma q u)_{\nu,m}(\cdot) e^\gamma_{\nu,m}$$

in $\mathcal{S}'(\mathbf{R}^n, E)$ with

$$\lambda^\gamma_{\nu,m} = \begin{cases} \lambda_{\nu,m}\|\eta^\gamma_{\nu,m}\|_E & ,\nu \geq \mu \\ 0 & ,\nu < \mu \end{cases} \quad \text{and} \quad e^\gamma_{\nu,m} = \begin{cases} \frac{\eta^\gamma_{\nu,m}}{\|\eta^\gamma_{\nu,m}\|_E} & ,\nu \geq \mu \\ 0 & ,\nu < \mu \end{cases} .$$

By (7) and the observations on the dependence of μ in (13) we find

$$2^{\mu|\gamma|}\|\lambda^\gamma|f_{p,q}\| \leq C' 2^{\mu\delta_1}\|f|F^s_{p,q}(\mathbf{R}^n, E)\|, \quad \gamma \in \mathbf{N}^n_0$$

with C' and δ_1 independent of μ and γ for $\mu \geq \varkappa_0$. □

Now we extend this theorem to more general atoms.

Theorem 3.11 *Let $K \in \mathbf{N}_0$ with $K \geq \lfloor s \rfloor + 1$.*
(i) Let $1 \leq p \leq \infty$, $1 \leq q \leq \infty$, and $s > 0$. Then there exists a $\varkappa > 0$ such that for all $\mu \geq \varkappa$ it is valid that $f \in \mathcal{S}'(\mathbf{R}^n, E)$ belongs to $B^s_{p,q}(\mathbf{R}^n, E)$ if and only if it can be represented by

$$f = \sum_{\gamma \in \mathbf{N}^n_0} \sum_{\nu=0}^{\infty} \sum_{m \in \mathbf{Z}^n} \varrho^\gamma_{\nu,m} a^\gamma_{\nu,m}(\cdot) e^\gamma_{\nu,m}$$

in $\mathcal{S}'(\mathbf{R}^n, E)$. Here $a^\gamma_{\nu,m}$ are scalar $(s,p)_K$-atoms, $e^\gamma_{\nu,m} \in U_E$ and

$$\sup_\gamma 2^{\mu|\gamma|}\|\varrho^\gamma|b_{p,q}\| < \infty.$$

Furthermore, we have in the sense of equivalence of norms

$$\|f|B^s_{p,q}(\mathbf{R}^n, E)\| \sim \inf \sup_\gamma 2^{\mu|\gamma|}\|\varrho^\gamma|b_{p,q}\|,$$

where the infimum on the right-hand side is taken over all admissible representations of f.
(ii) Let $1 \leq p < \infty$, $1 \leq q \leq \infty$, and $s > 0$. Then there exists a $\varkappa > 0$ such that for all $\mu \geq \varkappa$ it is valid that $f \in \mathcal{S}'(\mathbf{R}^n, E)$ belongs to $F^s_{p,q}(\mathbf{R}^n, E)$ if and only if it can be represented by

$$f = \sum_{\gamma \in \mathbf{N}^n_0} \sum_{\nu=0}^{\infty} \sum_{m \in \mathbf{Z}^n} \varrho^\gamma_{\nu,m} a^\gamma_{\nu,m}(\cdot) e^\gamma_{\nu,m}$$

in $\mathcal{S}'(\mathbf{R}^n, E)$. Here $a^\gamma_{\nu,m}$ are scalar $(s,p)_K$-atoms, $e^\gamma_{\nu,m} \in U_E$ and

$$\sup_\gamma 2^{\mu|\gamma|}\|\varrho^\gamma|f_{p,q}\| < \infty.$$

Furthermore, we have in the sense of equivalence of norms

$$\|f|F^s_{p,q}(\mathbf{R}^n, E)\| \sim \inf \sup_\gamma 2^{\mu|\gamma|}\|\varrho^\gamma|f_{p,q}\|,$$

where the infimum on the right-hand side is taken over all admissible representations of f.

4 Traces on hyperplanes

We shall follow two different strategies. First we characterize the trace $\mathrm{tr}\,(B^s_{p,q}(\mathbf{R}^n, E))$ of the Besov spaces. Here we use the techniques from Fourier analysis. Those will allow us to describe these trace spaces and at the same time to construct an universal extension operator.

Then we use the techniques of atomic decompositions for vector-valued function spaces. The starting point is a result of Frazier and Jawerth [16, 17] in the scalar case which shows that $\mathrm{tr}\,(F^s_{p,q}(\mathbf{R}^n)) = \mathrm{tr}\,(F^s_{p,p}(\mathbf{R}^n)) = \mathrm{tr}\,(B^s_{p,p}(\mathbf{R}^n))$. Based on the subatomic decompositions we are able to establish the vector-valued analogue. Having this at hand we deduce the trace space of the Sobolev spaces by a sandwich argument.

4.1 Definition and existence of the trace

We follow S. M. Nikol'skij [36, 37]. Here we restrict ourselves to the trace of a function $f(x',t)$, $x' \in \mathbf{R}^{n-1}$, $t \in \mathbf{R}$, to $\mathbf{R}^{n-1}$ (identified with $\{(x',0):\ x' \in \mathbf{R}^{n-1}\}$).

Definition 4.1 Let $f:\ \mathbf{R}^n \mapsto E$ be strongly measurable. The E-valued function g is called trace of f if the following is satisfied: There exist a function $\widetilde{f}:\ \mathbf{R}^n \mapsto E$, a number p, $1 \le p \le \infty$, and a positive number δ such that

- $\widetilde{f} = f$ a.e. (with respect to the Lebesgue measure in $\mathbf{R}^n$);
- $\widetilde{f}(\cdot,t) \in L_p(\mathbf{R}^{n-1},E)$ for all $|t| < \delta$;
- $\widetilde{f}(x',0) = g(x')$ a.e. (with respect to the Lebesgue measure in $\mathbf{R}^{n-1}$);
- $\lim_{t\to 0} \|\, \widetilde{f}(\cdot,t) - g\,|L_p(\mathbf{R}^{n-1},E)\| = 0\,.$

Remark 4.2 The definition of the trace does not depend on $\widetilde{f}$, p, and δ. This means, let us assume that there exist $\widetilde{f}_i, p_i, \delta_i, g_i$, $i = 1,2$, such that the conditions given in the definition are satisfied. We claim $g_1 = g_2$ a.e. in $\mathbf{R}^{n-1}$. For simplicity we assume $p_1 \le p_2$. Consider an arbitrary open and bounded set $\Omega \subset \mathbf{R}^{n-1}$. Then

$$\begin{aligned} \|\, g_1 - g_2\,|L_{p_1}(\Omega,E)\| \le \|\, g_1 - \widetilde{f}_1(\cdot,t)\,|L_{p_1}(\Omega,E)\| + \\ \|\widetilde{f}_1(\cdot,t) - \widetilde{f}_2(\cdot,t)|L_{p_1}(\Omega,E)\| + |\Omega|^{\frac{1}{p_1}-\frac{1}{p_2}}\, \|g_2 - \widetilde{f}_2(\cdot,t)|L_{p_2}(\Omega,E)\|\,. \end{aligned} \tag{14}$$

The equivalence of $\widetilde{f}_1$ and $\widetilde{f}_2$ implies

$$\int_{-\delta}^{\delta}\int_{\Omega} |\, \widetilde{f}_1(x',t) - \widetilde{f}_2(x',t)\,|^{p_1}\, dx'\, dt = 0$$

and hence, by Fubini's theorem for Bochner integrals,

$$\int_{\Omega} |\, \widetilde{f}_1(x',t) - \widetilde{f}_2(x',t)\,|^{p_1}\, dx' = 0 \tag{15}$$

for almost all $|t| < \delta$. We select a sequence $\{t_k\}_{k=1}^\infty$, $\lim_{k\to\infty} t_k = 0$, such that (15) is true for these t_k. On the right-hand side of (14) (with $t = t_k$) the first and the last term tend to zero when k tends to infinity and the second term always vanishes. So the left-hand side must vanish, too.

Remark 4.3 We define $\mathrm{tr}\, f := g$ if the trace exists. Then $\{f\ :\ \exists\ \mathrm{tr}\, f\}$ is a linear subset of the set of strongly measurable E-valued functions and tr is a linear operator. Furthermore, if f is continuous on $\mathbf{R}^{n-1} \times (-\delta,\delta)$ for some $\delta > 0$, then the trace of f exists and coincides with the pointwise restriction of f to $\{(x',0):\ x' \in \mathbf{R}^{n-1}\}$.

Proposition 4.4 *Let $1 \le p \le \infty$. Let $\{\varphi_j\}_{j=0}^\infty \in \Phi_{A,B}(\mathbf{R}^n)$. Then the trace of $f \in B^{1/p}_{p,1}(\mathbf{R}^n,E)$ exists and*

$$\mathrm{tr}\, f = \sum_{j=0}^{\infty} \mathcal{F}^{-1}[\varphi_j\, \mathcal{F} f](\cdot,0) \qquad (\textit{convergence in} \quad L_p(\mathbf{R}^{n-1},E))\,. \tag{16}$$

To prove the proposition we need a technical lemma.

Lemma 4.5 *Let $1 \le p \le \infty$. Let $g \in L_p(\mathbf{R}^n,E)$ such that $\mathrm{supp}\, \mathcal{F} g \subset [-1,1]^n$. Then there exists a constant c (independent of g) such that*

$$\|\, g(\cdot,0)\,|L_p(\mathbf{R}^{n-1},E)\| \le c\, \|\, g\,|L_p(\mathbf{R}^n,E)\|\,. \tag{17}$$

Proof. By assumption g is a strongly continuous function on $\mathbf{R}^n$, so the trace $g(x',0)$ is defined pointwise. Let $\psi \in C_0^\infty(\mathbf{R})$ such that $\psi(t) = 1$ if $|t| \le 1$ and $\mathrm{supp}\, \psi \subset [-2,2]$. We put $\Psi(x) = \psi(x_1) \cdot \ldots \cdot \psi(x_n)$. Then

$$g(x',0) = c \int_{\mathbf{R}^{n-1}} \left(\int_{-\infty}^{\infty} g(x'-y',-y_n)\, \mathcal{F}^{-1}\Psi(y',y_n)\, dy_n \right) dy'\,,$$

Let $p < \infty$ and let $1/p' = 1 - 1/p$. It follows

$$\| g(x',0) \|_E \le \int_{\mathbf{R}^{n-1}} \left[\int_{-\infty}^{\infty} \| g(x'-y',-y_n) \|_E^p \, dy_n \right]^{\frac{1}{p}} \cdot \left[\int_{-\infty}^{\infty} | \mathcal{F}^{-1}\Psi(y',y_n) |^{p'} dy_n \right]^{\frac{1}{p'}} dy'$$

by Hölder's inequality. Taking the L_p-norm with respect to x' implies

$$\| g(\cdot,0) \, | L_p(\mathbf{R}^{n-1},E) \| \le \| g \, | L_p(\mathbf{R}^n,E) \| \, \times \int_{\mathbf{R}^{n-1}} \left[\int_{-\infty}^{\infty} | \mathcal{F}^{-1}\Psi(y',y_n) |^{p'} \, dy_n \right]^{1/p'} dy'$$

by Minkowski's inequality. The second factor on the right-hand side is finite. The modification for $p = \infty$ is obvious. The proof is complete. □

Remark 4.6 A homogeneity argument in (17) leads to

$$\| g(\cdot,0) \, | L_p(\mathbf{R}^{n-1},E) \| \le c \, 2^{j/p} \, \| g \, | L_p(\mathbf{R}^n,E) \| \, , \tag{18}$$

if $\operatorname{supp} \mathcal{F}g \subset [-2^j, 2^j]^n$, where c neither depends on $j = 0, 1, \ldots$ nor on g. By a shift of the function the support of the Fourier transform is not changed. This implies that (18) remains true if one replaces 0 by any fixed t in $\mathbf{R}$ without affecting the constant c.

P r o o f. (of Proposition 4.4) We claim that

$$\widetilde{f}(x',t) = \sum_{j=0}^{\infty} \mathcal{F}^{-1}[\varphi_j \, \mathcal{F}f](x',t) = \sum_{j=0}^{\infty} f_j(x',t)$$

satisfies all the conditions of the definition. First of all, observe that

$$\lim_{N\to\infty} \| f - \sum_{j=0}^{N} f_j \, | L_p(\mathbf{R}^n,E) \| = 0$$

if $f \in B^{1/p}_{p,1}(\mathbf{R}^n,E)$. Hence $\widetilde{f} = f$ a.e. on $\mathbf{R}^n$. Making use of the triangle inequality and (18) we obtain

$$\begin{aligned} \| \widetilde{f}(\cdot,t) \, | L_p(\mathbf{R}^{n-1},E) \| &\le \sum_{j=0}^{\infty} \| \mathcal{F}^{-1}[\varphi_j \, \mathcal{F}f](\cdot,t) \, | L_p(\mathbf{R}^{n-1},E) \| \\ &\le c \sum_{j=0}^{\infty} 2^{j/p} \, \| \mathcal{F}^{-1}[\varphi_j \, \mathcal{F}f] \, | L_p(\mathbf{R}^n,E) \| \\ &= c \, \| f \, | B^{1/p}_{p,1}(\mathbf{R}^n,E) \| \, , \end{aligned}$$

where c does not depend on $t \in \mathbf{R}$. It remains to prove the last condition in the definition. Let $\varepsilon > 0$ be given. Because of

$$\left\| \sum_{j=N}^{\infty} \Big(\mathcal{F}^{-1}[\varphi_j \, \mathcal{F}f](\cdot,t) - \mathcal{F}^{-1}[\varphi_j \, \mathcal{F}f](\cdot,0) \Big) \Big| L_p(\mathbf{R}^{n-1},E) \right\| \le 2 \sum_{j=N}^{\infty} 2^{j/p} \| \mathcal{F}^{-1}[\varphi_j \, \mathcal{F}f] \, | L_p(\mathbf{R}^n,E) \|,$$

cf. (18), we may choose a natural number N such that

$$\| \sum_{j=N}^{\infty} \mathcal{F}^{-1}[\varphi_j \, \mathcal{F}f](\cdot,t) - \mathcal{F}^{-1}[\varphi_j \, \mathcal{F}f](\cdot,0) \, | L_p(\mathbf{R}^{n-1},E) \| \le \frac{1}{2} \, \varepsilon \, .$$

Having fixed this N we consider the difference for $j < N$. Once again we apply (18) and afterwards Lemma 2.3. This yields

$$\begin{aligned}\| \mathcal{F}^{-1}[\varphi_j \, \mathcal{F}f](\cdot,t) - \mathcal{F}^{-1}[\varphi_j \, \mathcal{F}f](\cdot,0) \, |L_p(\mathbf{R}^{n-1},E)\| \\ \le c\, 2^{j/p}\, |t| \cdot \Big\| \sup_{|y|<2^{-j}} \|\nabla \mathcal{F}^{-1}[\varphi_j \, \mathcal{F}f](\cdot,y)\|_E \, \Big| L_p(\mathbf{R}^n) \Big\| \\ \le c'\, 2^{j/p}\, |t|\, 2^j \cdot \Big\| \Big(M \|f_j\|_E^r \Big)^{1/r} \Big| L_p(\mathbf{R}^n) \Big\| \\ \le C\, 2^{j/p}\, |t|\, 2^j \cdot \| \, f_j \, |L_p(\mathbf{R}^n,E)\|\end{aligned}$$

with $|t| < 2^{-N}$ and $0 < r < p$. Choosing $|t| < \varepsilon\, C^{-1}\, 2^{-N}\, \| \, f \, |B^{1/p}_{p,1}(\mathbf{R}^n,E)\|^{-1}$ (where C denotes the constant from the preceding inequality) we find

$$\begin{aligned}\Big\| \sum_{j=0}^{\infty} \Big(\mathcal{F}^{-1}[\varphi_j \, \mathcal{F}f](\cdot,t) - \mathcal{F}^{-1}[\varphi_j \, \mathcal{F}f](\cdot,0) \Big) \Big| L_p(\mathbf{R}^{n-1},E) \Big\| \\ \le \frac{1}{2}\, \varepsilon + C\, |t| \sum_{j=0}^{N-1} 2^{j(1+1/p)}\, \| \, f_j \, |L_p(\mathbf{R}^n,E)\| \; \le \varepsilon \, .\end{aligned}$$

This proves our claim. □

Remark 4.7 Provided that $f \in B^{1/p}_{p,1}(\mathbf{R}^n,E)$, we shall always work with the identity (16).

Remark 4.8 There are several further concepts to define a trace. Most elegant but less transparent is the approach of Peetre [38], see also Johnsen [28] and Farkas, Johnsen, and Sickel [13], to deal with the trace on subspaces of $C_b(\mathbf{R},\mathcal{D}'(\mathbf{R}^{n-1},E))$. Amann [1] introduced the trace on subspaces of $W^{1,\ell oc}_1(\mathbf{R}^n,E)$. Johnson and Wallin [29] prefer to work with a pointwise definition. They replace the locally integrable function f by

$$\overline{f}(x) := \lim_{r\to 0} \frac{1}{|B(x,r)|} \int_{B(x,r)} f(y)\, dy$$

and investigate the pointwise trace of $\overline{f}$ instead of f itself. A further approach makes use of the immediate definition of the trace for smooth functions and argues by means of continuous extensions, cf. Triebel [53, 54, 56]. All these different methods coincide on subspaces of $B^{1/p}_{p,1}(\mathbf{R}^n,E)$.

4.2 The trace space of a Besov space

Now we are ready for a first important result in this section.

Theorem 4.9 *Suppose $n \ge 2$. Let $1 \le p \le \infty$, $1 \le q \le \infty$, and $s > 1/p$. Then the restriction of* tr *to $B^s_{p,q}(\mathbf{R}^n,E)$ is a bounded mapping onto $B^{s-1/p}_{p,q}(\mathbf{R}^{n-1},E)$. Furthermore, there is a universal linear and bounded extension operator* ext *mapping $B^{s-1/p}_{p,q}(\mathbf{R}^{n-1},E)$ into $B^s_{p,q}(\mathbf{R}^n,E)$ such that* $\mathrm{tr}\circ\mathrm{ext} = \mathrm{id}$ *on $B^{s-1/p}_{p,q}(\mathbf{R}^{n-1},E)$.*

Remark 4.10 Here *universal* means that the extension does not depend on the parameters s, p, and q.

Remark 4.11 A first proof of Theorem 4.9 in the scalar case and for $q = \infty$ has been given by Nikol'skij [36] at the beginning of the fifties. Later on there have been given proofs by different techniques including investigations of the existence of a linear extension operator. We refer to Peetre [39], Burenkov and Gol'dman [8], Gol'dman [21], Jawerth [27], Frazier and Jawerth [16], Triebel [53, 54], Johnson and Wallin [29], and the references given there. The more general anisotropic vector-valued version of Theorem 4.9 has been given recently in Amann [4]. There a proof is sketched based on our earlier papers [13] and [49]. Moreover, traces on certain manifolds are considered in [4].

P r o o f. *Step 1.* Preparation. Let $f \in \mathcal{S}(\mathbf{R}^n, E)$. Then $g(x') = f(x', 0) \in \mathcal{S}(\mathbf{R}^{n-1}, E)$. If $\operatorname{supp} \mathcal{F}_n f \subset \{\xi : |\xi| \le \lambda\}$, then also g satisfies $\operatorname{supp} \mathcal{F}_{n-1} g \subset \{\xi' : |\xi'| \le \lambda\}$. To see this one can argue as follows. We put

$$h(\xi', x_n) = \int_{\mathbf{R}^{n-1}} f(x', x_n)\, e^{-ix'\xi'}\, dx' .$$

For $L_1(\mathbf{R}^n, E)$-functions the Fourier transform may be written as a Bochner integral. This yields

$$0 = \int_{\mathbf{R}} h(\xi', x_n)\, e^{-ix_n \xi_n}\, dx_n$$

for all $|\xi'| > \lambda$. Hence, by the uniqueness of the Fourier transform on $\mathcal{S}(\mathbf{R}, E)$ we conclude $h(\xi', \cdot) \equiv 0$. In particular $h(\xi', 0) = 0$ for all $|\xi'| > \lambda$.

Step 2. Now we are going to extend this from $\mathcal{S}(\mathbf{R}^n, E)$ to $C_{ub}(\mathbf{R}^n, E)$. Here we use a standard procedure. Denote by ω a smooth function supported in the unit ball with $\int w(x)\, dx = 1$. Further we put $\omega_h(x) = h^{-n}\, \omega(x/h)$, $h > 0$. Let $f \in C_b^\infty(\mathbf{R}^n, E)$ such that $\operatorname{supp} \mathcal{F}_n f \subset \{\xi : |\xi| \le \lambda\}$. Then we define an approximation of f in $\mathcal{S}'(\mathbf{R}^n, E)$ by means of

$$f^h(x) := (2\pi)^{n/2}\, f(x)\, (\mathcal{F}^{-1}\omega_h)(x) = \mathcal{F}^{-1}[\mathcal{F} f * \omega_h](x) .$$

Then $f^h \in \mathcal{S}(\mathbf{R}^n, E)$ and $\operatorname{supp} \mathcal{F} f^h \subset \{\xi : |\xi| \le \lambda + h\}$. Furthermore, because of

$$\lim_{h \to 0} (2\pi)^{n/2} \Big(\mathcal{F}^{-1}\omega\Big)(hx) = \lim_{h\to 0} \int e^{ihxy}\, \omega(y)\, dy = 1$$

we conclude

$$f^h(x', 0) \longrightarrow f(x', 0) \qquad \text{if} \quad h \to 0 \tag{19}$$

in the pointwise sense and in the sense of $\mathcal{S}'(\mathbf{R}^{n-1}, E)$, too. Step 1 applied to f^h yields

$$\operatorname{supp} (\mathcal{F}_{n-1} f^h(x', 0)) \subset \{\xi' : |\xi'| \le \lambda + h\} .$$

Consequently, by (19) and the continuity of the Fourier transform as an operator in $\mathcal{S}'(\mathbf{R}^{n-1}, E)$

$$\begin{aligned} \int_{\mathbf{R}^{n-1}} f(x', 0)\, \varphi(x')\, dx' &= \lim_{h\to 0} \int_{\mathbf{R}^{n-1}} f^h(x', 0)\, \varphi(x')\, dx' \\ &= \lim_{h \to 0} \int_{\mathbf{R}^{n-1}} \mathcal{F}_{n-1}(f^h(x', 0))(\xi')\, \mathcal{F}_{n-1}\varphi(\xi')\, d\xi' \\ &= 0 , \end{aligned}$$

if $\varphi \in \mathcal{S}(\mathbf{R}^{n-1})$ and

$$\Big|\operatorname{supp} \mathcal{F}_{n-1}\varphi \cap \{\xi' : |\xi'| \le \lambda\}\Big| = 0 .$$

This proves $\operatorname{supp} (\mathcal{F}_{n-1} f(x', 0)) \subset \{\xi : |\xi| \le \lambda\}$.

Step 3. We collect the consequences of Step 1 and 2 we are interested in. Suppose $\{\varphi_j\}_{j=0}^\infty \in \Phi_{1,3/2}(\mathbf{R}^n)$ and $\{\phi_j\}_{j=0}^\infty \in \Phi_{1,3/2}(\mathbf{R}^{n-1})$. Let $f \in L_p(\mathbf{R}^n, E)$. We put $g_j(x') = f_j(x', 0)$. For $p < \infty$ the observations made in Step 1 and a density argument show that for $\ell \ge 2$

$$\phi_\ell(\xi')\, \mathcal{F}_{n-1} g_j(\xi') \equiv 0 , \qquad j = 0, 1, \ldots, \ell - 2 ,$$

holds in $L_p(\mathbf{R}^{n-1}, E)$. Analogously we may argue for $f \in C_{ub}(\mathbf{R}^n, E)$ using Step 2 now.

Step 4. Putting $f_{-1} \equiv 0$, employing Step 3, and using the triangle inequality we find

$$\begin{aligned}\| \mathcal{F}_{n-1}^{-1}[\phi_\ell(\xi')\, \mathcal{F}_{n-1} f(\cdot\,,0)(\xi')](\cdot)\, |L_p(\mathbf{R}^{n-1},E)\| & \\ \le \sum_{j=\ell-1}^{\infty} \| \mathcal{F}_{n-1}^{-1}[\phi_\ell(\xi')\, \mathcal{F}_{n-1} f_j(\cdot,0)(\xi')](\cdot)\, |L_p(\mathbf{R}^{n-1},E)\| & \\ \le c \sum_{j=0}^{\infty} \| f_{j+\ell-1}(\cdot,0)\, |L_p(\mathbf{R}^{n-1},E)\|\,, &\end{aligned}$$

where we used a convolution argument in the last step. The latter inequality and (18) lead to

$$\begin{aligned}\| f(\cdot,0)|B^s_{p,q}(\mathbf{R}^{n-1},E)\| &= \Big(\sum_{\ell=0}^{\infty} 2^{\ell sq}\, \| \mathcal{F}_{n-1}^{-1}[\phi_\ell(\xi')\, \mathcal{F}_{n-1} f(\cdot,0)(\xi')](\cdot)\, |L_p(\mathbf{R}^{n-1},E)\|^q\Big)^{1/q} \\ &\le c \sum_{j=0}^{\infty} \Big\{ \sum_{\ell=0}^{\infty} 2^{\ell sq}\, \| f_{j+\ell-1}(\cdot,0)\, |L_p(\mathbf{R}^{n-1},E)\|^q \Big\}^{1/q} \\ &\le c \sum_{j=0}^{\infty} \Big\{ \sum_{\ell=0}^{\infty} 2^{\ell sq} \Big(2^{(j+\ell-1)/p} \| f_{j+\ell-1}\, |L_p(\mathbf{R}^{n},E)\| \Big)^q \Big\}^{1/q} \\ &\le c \sum_{j=0}^{\infty} 2^{-(j-1)s} \Big\{ \sum_{\ell=0}^{\infty} 2^{(j+\ell-1)(s+1/p)q}\, \| f_{j+\ell-1}\, |L_p(\mathbf{R}^{n},E)\|^q \Big\}^{1/q} \\ &\le C\, \| f\, |B^{s+1/p}_{p,q}(\mathbf{R}^{n},E)\|\end{aligned}$$

as far as $s > 0$. This proves that under the restrictions of the theorem the mapping tr is into.

Step 5. The surjectivity follows from Proposition 4.12 below. □

One of the advantages of the Fourier-analytic approach to the trace problem consists in the possibility to work with an universal extension procedure. Here we make use of a construction introduced in the scalar case by Triebel, cf. [53, 2.7.2/(26)]. Let $\{\varrho_j\}_{j=0}^\infty \in \Phi_{1,2}(\mathbf{R})$ such that $2^{-j}\, \mathcal{F}^{-1}[\varrho_j](0) = 1$ for all j and let $\{\phi_j\}_{j=0}^\infty \in \Phi_{1,2}(\mathbf{R}^{n-1})$. For $f \in \mathcal{S}'(\mathbf{R}^{n-1},E)$ we put

$$\mathrm{ext}\, f(x',x_n) = \sum_{j=0}^{\infty} 2^{-j}\, \mathcal{F}_1^{-1}[\varrho_j](x_n)\, \mathcal{F}_{n-1}^{-1}[\phi_j(\xi')\, \mathcal{F}_{n-1} f(\xi')](x')\,.$$

The basic observation consists in the identity

$$\mathcal{F}^{-1}[\varphi_\ell\, \mathcal{F}\mathrm{ext}\, f](x) = \sum_{j=-1}^{1} 2^{-(j+\ell)}\, \mathcal{F}^{-1}\Big[\varphi_\ell\, \mathcal{F}\Big(\mathcal{F}_1^{-1}[\varrho_{j+\ell}](y_n)\, f_{j+\ell}(y')\Big)\Big](x) \tag{20}$$

for any system $\{\varphi_\ell\}_\ell \in \Phi_{1,2}(\mathbf{R}^n)$.

Proposition 4.12 *Let $1 \le p,q \le \infty$ and $s > 1/p$. Then we have*

$$\mathrm{ext} \in \mathcal{L}\Big(B^{s-1/p}_{p,q}(\mathbf{R}^{n-1},E), B^s_{p,q}(\mathbf{R}^n,E)\Big)\,,$$

and if $1 < p < \infty$ and $1 < q \le \infty$ also

$$\mathrm{ext} \in \mathcal{L}\Big(B^{s-1/p}_{p,p}(\mathbf{R}^{n-1},E), F^s_{p,q}(\mathbf{R}^n,E)\Big)\,.$$

P r o o f. *Step 1.* We prove the first assertion. We have $\mathrm{tr}\circ\mathrm{ext} = \mathrm{id}$ (convergence in $\mathcal{S}'(\mathbf{R}^n, E)$), say, for $f \in \mathcal{S}(\mathbf{R}^{n-1}, E)$ at the moment. Next we shall prove boundedness for ext. First, observe

$$\begin{aligned}
&2^{-(\ell+j)}\|\mathcal{F}_n^{-1}[\varphi_\ell(\xi)\,\phi_{\ell+j}(\xi')\,\mathcal{F}_{n-1}f(\xi')\ \varrho_{\ell+j}(\xi_n)](\cdot)\,|L_p(\mathbf{R}^n,E)\| \\
&\qquad \le c\,2^{-(\ell+j)}\|\mathcal{F}_{n-1}^{-1}[\phi_{\ell+j}(\xi')\mathcal{F}_{n-1}f(\xi')]|L_p(\mathbf{R}^{n-1},E)\|\cdot\|\mathcal{F}_1^{-1}[\varrho_{\ell+j}](\cdot)\,|L_p(\mathbf{R})\| \\
&\qquad \le c\,2^{-(\ell+j)/p}\|\,\mathcal{F}_{n-1}^{-1}[\phi_{\ell+j}(\xi')\mathcal{F}_{n-1}f(\xi')]\,|L_p(\mathbf{R}^{n-1},E)\|,
\end{aligned}$$

where we have applied again a convolution argument. In view of (20) this gives

$$\begin{aligned}
\|\mathrm{ext}\,f|B^s_{p,q}(\mathbf{R}^n,E)\| &= \left(\sum_{\ell=0}^\infty 2^{\ell sq}\,\|\mathcal{F}_n^{-1}[\varphi_\ell(\xi)\mathcal{F}_n\mathrm{ext}\,f(\xi)](\cdot)|L_p(\mathbf{R}^n,E)\|^q\right)^{1/q} \\
&\le c\max_{j=-1,0,1}\left(\sum_{\ell=0}^\infty 2^{\ell sq}\,2^{-\ell/p}\|\,f_{j+\ell}\,|L_p(\mathbf{R}^{n-1},E)\|^q\right)^{1/q} \\
&\le C\,\|\,f\,|B^{s-1/p}_{p,q}(\mathbf{R}^{n-1},E)\|\,.
\end{aligned}$$

This proves $\mathrm{ext} \in \mathcal{L}(B^{s-1/p}_{p,q}(\mathbf{R}^{n-1},E), B^s_{p,q}(\mathbf{R}^n,E))$ without any restriction on s, p, and q. From this and $\mathrm{tr}\circ\mathrm{ext} = \mathrm{id}$ we conclude that tr is in fact a mapping onto.

We assumed that the function f belongs to $\mathcal{S}(\mathbf{R}^{n-1}, E)$. This is a dense set in the Besov spaces if $\max(p,q) < \infty$. Hence, together with the proven continuity of ext, we have finished our proof for those values of p and q. Now, let $\max(p,q) = \infty$. Then we apply the Fatou property of the Besov spaces, cf. Proposition 2.18, together with

$$\lim_{M\to\infty}\sum_{j=0}^M \mathcal{F}^{-1}[\phi_j\,\mathcal{F}f] = f \qquad (\text{convergence in the sense of } \mathcal{S}'(\mathbf{R}^{n-1},E))\,.$$

Note that all the estimates given above remain true if f is replaced by $\sum_{j=0}^M \mathcal{F}^{-1}[\phi_j\,\mathcal{F}f]$ with constants independent of M and f. This completes the proof (i).

Step 2. Proof of (ii). In view of the arguments at the end of Step 1 it remains to prove the boundedness. Without loss of generality we may assume $1 < q < p < \infty$. From Lemma 2.4 (using $\varphi \in \mathcal{S}(\mathbf{R}^n)$) it follows

$$\|\mathcal{F}^{-1}[\varphi_\ell\,\mathcal{F}\mathrm{ext}\,f](x)\|_E \le \sum_{j=-1}^1 2^{-(j+\ell)}\,M\Big(|\mathcal{F}_1^{-1}[\varrho_{j+\ell}](y_n)|\,\|\,f_{j+\ell}(y')\|_E\Big)(x)\,,$$

where c does not depend on f, ℓ, and x. This together with the vector-valued Hardy-Littlewood maximal inequality, cf. [14], and the triangle inequality in $L_{p/q}(\mathbf{R}^{n-1})$ yields

$$\begin{aligned}
&\|\mathrm{ext}\,f\,|F^{s+1/p}_{p,q}\| \\
&\quad\le c\sum_{j=-1}^1\left\|\left(\sum_{\ell=0}^\infty\left\{\|\,2^{(j+\ell)(s+1/p-1)}|\mathcal{F}_1^{-1}[\varrho_{j+\ell}](y_n)|\,f_{j+\ell}(y')\|_E\right\}^q\right)^{1/q}\right|L_p(\mathbf{R}^n)\Bigg\| \\
&\quad= c\sum_{j=-1}^1\left(\int_{-\infty}^\infty\left\|\sum_{\ell=0}^\infty\{\|\ldots\|_E\}^q\right|L_{p/q}(\mathbf{R}^{n-1})\Bigg\|^{p/q}dy_n\right)^{1/p} \\
&\quad\le c\sum_{j=-1}^1\left(\int_{-\infty}^\infty\left(\sum_{\ell=0}^\infty 2^{-(j+\ell)q}\left|\mathcal{F}_1^{-1}[\varrho_{j+\ell}](y_n)\right|^q\,b^q_{j+\ell}\right)^{p/q}dy_n\right)^{1/p},
\end{aligned}\tag{21}$$

where we used the abbreviation

$$b_k = 2^{k(s+1/p)}\|\,f_k\,|L_p(\mathbf{R}^{n-1},E)\|\,.$$

It is only a technical matter to show that the right-hand side in (21) is bounded from above by a constant times $\| f \,|B^s_{p,p}(\mathbf{R}^{n-1},E)\|$, we refer to [53, Theorem 2.7.2, Step 6] for details. This proves the boundedness of ext in this case, too. □

Remark 4.13 In Theorem 4.9 we stated that the extension operator is universal which means that the definition does not depend on the parameters of the spaces. However, it is universal also in a wider sense. Suppose E_1 and E_2 are Banach spaces such that $E_1 \hookrightarrow E_2$. Then $B^s_{p,q}(\mathbf{R}^n,E_1) \hookrightarrow B^s_{p,q}(\mathbf{R}^n,E_2)$, cf. Proposition 2.13. From this observation it becomes clear that $\mathrm{ext}_{\,E_1}$ and $\mathrm{ext}_{\,E_2}$ coincide on $B^s_{p,q}(\mathbf{R}^n,E_1)$.

Remark 4.14 In 1998 H. Amann posed the following question:
Let $E_1 \hookrightarrow E_2$. What can be said about the trace of the intersection of $B^s_{p,q}(\mathbf{R}^n,E_1)$ and $B^{s+1}_{p,q}(\mathbf{R}^n,E_2)$?
Applying Theorem 4.9 and the preceding remark we obtain

$$\begin{aligned} B :=\ & \mathrm{tr}\,(B^s_{p,q}(\mathbf{R}^n,E_1) \cap B^{s+1}_{p,q}(\mathbf{R}^n,E_2)) \\ =\ & B^{s-1/p}_{p,q}(\mathbf{R}^{n-1},E_1) \cap B^{s+1-1/p}_{p,q}(\mathbf{R}^{n-1},E_2)\,, \end{aligned}$$

provided that $s > 1/p$. Here $\mathrm{tr}\, A(\mathbf{R}^n) = B(\mathbf{R}^{n-1})$ means that $\mathrm{tr} \in \mathcal{L}(A,B)$, $\mathrm{ext} \in \mathcal{L}(B,A)$, and $\mathrm{tr} \circ \mathrm{ext} = \mathrm{id}$ on B.

4.3 The trace of Sobolev spaces

From the embeddings derived above we know

$$B^{m-1/p}_{p,1}(\mathbf{R}^n,E) \hookrightarrow \mathrm{tr}\,(W^m_p(\mathbf{R}^n,E)\,) \hookrightarrow B^{m-1/p}_{p,\infty}(\mathbf{R}^n,E)\,.$$

To characterize the trace of Sobolev spaces we make use of an argument due to Frazier and Jawerth [17] in the scalar case.

Lemma 4.15 *Let $1 \le p < \infty$, $1 \le q \le \infty$, and suppose $s > 1/p$. Then the trace of the elements of $F^s_{p,q}(\mathbf{R}^n,E)$ exists and*

$$\mathrm{tr}\,(F^s_{p,q}(\mathbf{R}^n,E)\,) = \mathrm{tr}\,(F^s_{p,p}(\mathbf{R}^n,E))\,.$$

The proof relies on the characterization of $F^s_{p,q}(\mathbf{R}^n,E)$ in terms of atoms. Let $s > 0$, $1 \le p < \infty$, and $1 \le q \le \infty$. By Theorem 3.11 there exists a positive number $\varkappa$ such that for $\mu \ge \varkappa$ the following characterization is true: An $f \in \mathcal{S}'(\mathbf{R}^n,E)$ belongs to $F^s_{p,q}(\mathbf{R}^n,E)$ if and only if it can be represented as

$$f = \sum_{\gamma\in\mathbf{N}_0^n} \sum_{\nu=0}^{\infty} \sum_{m\in\mathbf{Z}^n} \varrho^\gamma_{\nu,m}\, a^\gamma_{\nu,m}(\cdot)\, e^\gamma_{\nu,m}\,, \quad (\text{convergence in } \mathcal{S}'(\mathbf{R}^n,E))\,, \tag{22}$$

where $a^\gamma_{\nu,m}$ are $(s,p)_K$-atoms with respect to $Q_{\nu,m}$ for K sufficiently large, $e^\gamma_{\nu,m}$ belongs to the unit ball in E, and

$$\| f \,|F^s_{p,q}(\mathbf{R}^n,E)\,\|^{\clubsuit} = \sup_{\gamma\in\mathbf{N}_0^n} 2^{\mu\,|\gamma|}\, \|\,\varrho^\gamma\,|f_{p,q}\| < \infty\,. \tag{23}$$

Furthermore, taking the infimum with respect to all admissible representations in (23) one obtains an equivalent norm in $F^s_{p,q}(\mathbf{R}^n,E)$ for each admissible μ.

Here we select the function $\widetilde{f}$ in a different way than in case of the Besov spaces. It will be connected with (22). For given $t \in \mathbf{R}$ we denote by $I_{t,\nu}$ the subset of all lattice points $m \in \mathbf{Z}^n$ such that

$$|\,2\,Q_{\nu,m} \cap \{(x',t):\ x' \in \mathbf{R}^{n-1}\}\,| > 0\,,$$

(here $|\cdot|$ denotes the n-dimensional Lebesgue measure). Suppose that f is given by (22), then we define

$$\widetilde{f}(x',t) = \sum_{\gamma\in\mathbf{N}_0^n} \sum_{\nu=0}^{\infty} \sum_{m\in I_{t,\nu}} \varrho^\gamma_{\nu,m}\, a^\gamma_{\nu,m}(x',t)\, e^\gamma_{\nu,m}\,.$$

There is the following supplement to Proposition 4.4.

Lemma 4.16 *Let $1 \le p \le \infty$. Then the trace of $f \in B^{1/p}_{p,1}(\mathbf{R}^n, E)$ exists and*

$$\operatorname{tr} f = \sum_{\gamma \in \mathbf{N}_0^n} \sum_{\nu=0}^{\infty} \sum_{m \in I_{0,\nu}} \varrho^{\gamma}_{\nu,m} \, a^{\gamma}_{\nu,m}(x',0) \, e^{\gamma}_{\nu,m} \qquad (\textit{convergence in } L_p(\mathbf{R}^{n-1}, E))$$

holds for all representations (22) *with* $\| \, f \,|B^{1/p}_{p,1}(\mathbf{R}^n, E)\| < \infty$.

P r o o f. Besov spaces can be characterized by atoms in a similar way as the Lizorkin-Triebel spaces, see 3.11. It holds that $f \in B^{1/p}_{p,1}(\mathbf{R}^n, E)$ if and only if there is a representation of f as in (22), where now $a^{\gamma}_{\nu,m}$ are $(1/p, p)_K$-atoms such that for $\mu \ge \varkappa$

$$\| \, f \,|B^{1/p}_{p,1}(\mathbf{R}^n, E)\|^{\clubsuit} = \sup_{\gamma \in \mathbf{N}_0^n} 2^{\mu|\gamma|} \left(\sum_{\nu=0}^{\infty} \left(\sum_{m \in \mathbf{Z}^n} |\varrho^{\gamma}_{\nu,m}|^p \right)^{1/p} \right) < \infty \, ,$$

(the restrictions for $e^{\gamma}_{\nu,m}$ remain unchanged). The infimum taken with respect to all possible representations in (22) yields an equivalent norm on $B^{1/p}_{p,1}(\mathbf{R}^n, E)$. The $L_p(\mathbf{R}^n, E)$-convergence of such a representation follows easily. Now we concentrate on convergence in $L_p(\mathbf{R}^{n-1}, E)$. Let $t \in \mathbf{R}$ be given. Observe

$$\| \, a^{\gamma}_{\nu,m} \,|L_p(\mathbf{R}^{n-1}, E)\| \le 2^{n-1}$$

for all ν, m, and γ. Then, by using the localization of the supports of the atoms and $\| \, e^{\gamma}_{\nu,m} \, \|_E \le 1$ we conclude

$$\begin{aligned} \| \, \widetilde{f}(\cdot\,, t) \,|L_p(\mathbf{R}^{n-1}, E)\| &\le \sum_{\gamma \in \mathbf{N}_0^n} \sum_{\nu=0}^{\infty} \Big\| \sum_{m \in \mathbf{Z}^n} |\varrho^{\gamma}_{\nu,m}| \, |a^{\gamma}_{\nu,m}(\cdot\,, t)| \,|L_p(\mathbf{R}^{n-1}) \Big\| \\ &\le c \sum_{\gamma \in \mathbf{N}_0^n} \sum_{\nu=0}^{\infty} \left(\sum_{m \in \mathbf{Z}^n} |\varrho^{\gamma}_{\nu,m}|^p \right)^{1/p} \\ &\le C \, \| \, f \,|B^{1/p}_{p,1}(\mathbf{R}^n, E)\| \, , \end{aligned} \tag{24}$$

where C does not depend on f and t. By restricting the summation in an appropriate way convergence in $L_p(\mathbf{R}^{n-1}, E)$ follows for any t.

It remains to prove convergence of $\widetilde{f}(x', t) - \widetilde{f}(x', 0)$ if $t \to 0$. Here we shall use the abbreviation

$$C_\mu = \sum_{\gamma \in \mathbf{N}_0} 2^{-\mu|\gamma|} \, .$$

Given $\varepsilon > 0$ we select a natural number N such that

$$C \sum_{\gamma \in \mathbf{N}_0^n} \sum_{\nu=N}^{\infty} \left(\sum_{m \in \mathbf{Z}^n} |\varrho^{\gamma}_{\nu,m}|^p \right)^{1/p} \le C_\mu^{-1} \, \varepsilon/2$$

and C is the constant in (24). Making use of the Lipschitz continuity of the atoms, cf. (4), we derive

$$\begin{aligned} \sum_{\gamma \in \mathbf{N}_0^n} \sum_{\nu=0}^{N} \Big\| \sum_{m \in \mathbf{Z}^n} \varrho^{\gamma}_{\nu,m} \left(a^{\gamma}_{\nu,m}(x', t) - a^{\gamma}_{\nu,m}(x', 0) \right) e^{\gamma}_{\nu,m} \, \Big| L_p(\mathbf{R}^{n-1}) \Big\| \\ \le \sum_{\gamma \in \mathbf{N}_0^n} \sum_{\nu=0}^{N} |t| \, 2^{\nu} \left(\sum_{m \in \mathbf{Z}^n} |\varrho^{\gamma}_{\nu,m}|^p \right)^{1/p} \\ \le |t| \, 2^N \, \| \, f \,|B^{1/p}_{p,1}(\mathbf{R}^n, E)\| \, . \end{aligned}$$

For $|t| < C_\mu^{-1} \, 2^{-N-1} \, \varepsilon$ these estimates imply

$$\| \, \widetilde{f}(x', t) - \widetilde{f}(x', 0) \,|L_p(\mathbf{R}^{n-1}, E)\| < \varepsilon \, ,$$

which completes the proof. □

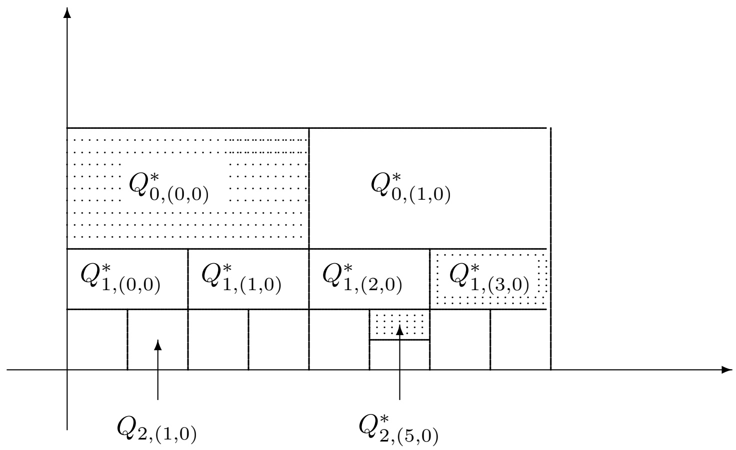

Fig. 1

P r o o f. (of Lemma 4.15) Let $q < \infty$. Now we are going to prove: If $f \in F^s_{p,\infty}(\mathbf{R}^n, E)$, then there exists a function $g \in F^s_{p,q}(\mathbf{R}^n, E)$ such that $\mathrm{tr}\, f = \mathrm{tr}\, g$.
Suppose that f is given by (22), then we define

$$g := \sum_{\gamma\in\mathbf{N}_0^n} \sum_{\nu=0}^{\infty} \sum_{m\in I_{0,\nu}} \varrho^\gamma_{\nu,m}\, a^\gamma_{\nu,m}(x)\, e^\gamma_{\nu,m}\,,$$

By construction the traces of f and g exist and coincide. This follows from Lemma 4.16 and the continuous embeddings

$$F^s_{p,q}(\mathbf{R}^n, E) \hookrightarrow F^s_{p,\infty}(\mathbf{R}^n, E) \hookrightarrow B^{1/p}_{p,1}(\mathbf{R}^n, E)\,.$$

It remains to prove $g \in F^s_{p,q}(\mathbf{R}^n, E)$. Let

$$Q^*_{\nu,m} = \Big\{ x \in \mathbf{R}^n :\ 2^{-\nu} m_i \le x_i < 2^{-\nu}(m_i + 1),\ i = 1, \ldots , n-1,$$
$$2^{-\nu}(m_n + 1/2) \le x_i < 2^{-\nu}(m_n + 1) \Big\}, \quad \nu \in \mathbf{N}_0, \quad m \in \mathbf{Z}^n\,.$$

Looking at the cubes $Q_{\nu,m}$ such that $m \in I_{0,\nu}$ it turns out that the rectangles $Q^*_{\nu,m}$ of those cubes are essentially disjoint, cf. Figure 1. More exactly,

$$|\, Q^*_{\nu_1,m_1} \cap Q^*_{\nu_2,m_2}\,| = 0\,, \qquad |\nu_1 - \nu_2| + |m_1 - m_2| > 0, \quad m_1 \in I_{0,\nu_1},\, m_2 \in I_{0,\nu_2}\,.$$

By a result of Frazier and Jawerth

$$\|\, \lambda \,|f_{p,q}\| \sim \Big\| \Big(\sum_{\nu=0}^{\infty} \sum_{m\in\mathbf{Z}^n} |\lambda_{\nu,m}\, \chi^{*,(p)}_{\nu,m}(x)|^q \Big)^{1/q} \Big| L_p(\mathbf{R}^n) \Big\|,$$

where $\chi^{*,(p)}_{\nu,m}(x)$ denotes the L_p-normalized characteristic function of $Q^*_{\nu,m}$, cf. [17, Cor. 5.6]. Define

$$\tau^\gamma_{\nu,m} := \begin{cases} \varrho^\gamma_{\nu,m} & \text{if} \quad m \in I_\nu\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

Because of the disjointness of the $Q^*_{\nu,m}$ we have

$$\begin{aligned}\| \tau^\gamma \,|f_{p,q}\| &\le c \bigg\| \Big(\sum_{\nu=0}^\infty \sum_{m\in I_\nu} |\varrho^\gamma_{\nu,m}\, \chi^{*,(p)}_{\nu,m}(x)|^q \Big)^{1/q} \Big| L_p(\mathbf{R}^n) \bigg\| \\ &= c \Big\| \sup_{\nu=0,1,\ldots} \sup_{m\in I_\nu} |\varrho^\gamma_{\nu,m}\, \chi^{*,(p)}_{\nu,m}(x)| \Big| L_p(\mathbf{R}^n) \Big\| \\ &\le c \,\| \varrho^\gamma \,|f_{p,\infty}\| \,.\end{aligned}$$

But this implies

$$2^{\mu|\gamma|}\, \| \tau^\gamma \,|f_{p,q}\| \le c\, 2^{\mu|\gamma|}\, \| \varrho^\gamma \,|f_{p,\infty}\| \le c \sup_\gamma 2^{\mu|\gamma|}\, \| \varrho^\gamma \,|f_{p,\infty}\| \le c \,\| f \,|F^s_{p,\infty}(\mathbf{R}^n,E)\|$$

for properly chosen ϱ^γ. Hence $g \in F^s_{p,q}(\mathbf{R}^n,E)$ and

$$\| g \,|F^s_{p,q}(\mathbf{R}^n,E)\,\|^{\clubsuit} \le c \,\| f \,|F^s_{p,\infty}(\mathbf{R}^n,E)\|^{\clubsuit} \,.$$

This proves the claim. □

As a consequence of Lemma 4.15 and Proposition 4.12 we obtain a characterization of the traces of $F^s_{p,q}(\mathbf{R}^n,E)$.

Theorem 4.17 *Suppose $n \ge 2$. Let $1 \le p < \infty$, $1 \le q \le \infty$, and $s > 1/p$. Then the restriction of* tr *to $F^s_{p,q}(\mathbf{R}^n,E)$ is a bounded mapping onto $B^{s-1/p}_{p,p}(\mathbf{R}^{n-1},E)$. Furthermore, there exists a universal linear and bounded extension operator* ext *mapping $B^{s-1/p}_{p,p}(\mathbf{R}^{n-1},E)$ into $F^s_{p,q}(\mathbf{R}^n,E)$ such that* $\mathrm{tr}\circ\mathrm{ext} = \mathrm{id}$ *on $B^{s-1/p}_{p,p}(\mathbf{R}^{n-1},E)$.*

Remark 4.18 Restricted to the scalar situation there are further proofs of this assertion. We refer to Nikol'skij [37] ($q = 2$), Jawerth [26, 27], Kalyabin [30], Gol'dman [22], and Triebel [53, 2.7.2], [54, 4.4]. Limiting cases are treated in Frazier and Jawerth [17], Triebel [54, 4.4.3], and Farkas, Johnsen, and Sickel [13].

By our sandwich argument, see Theorem 2.20, we finally deduce the main result of this section.

Theorem 4.19 *Let $1 < p < \infty$.*
(i) *Let $m \in \mathbf{N}$. Then the trace operator* tr *is a continuous mapping from $W^m_p(\mathbf{R}^n,E)$ onto $B^{m-1/p}_{p,p}(\mathbf{R}^{n-1},E)$. Moreover, there exists a universal extension operator* ext *belonging to $\mathcal{L}\Big(B^{m-1/p}_{p,p}(\mathbf{R}^{n-1},E), W^m_p(\mathbf{R}^n,E)\Big)$ such that* $\mathrm{tr}\circ\mathrm{ext} = \mathrm{id}$ *on $B^{m-1/p}_{p,p}(\mathbf{R}^{n-1},E)$.*
(ii) *Let $\sigma > 1/p$. Then the trace operator* tr *is a continuous mapping of $H^\sigma_p(\mathbf{R}^n,E)$ onto $B^{\sigma-1/p}_{p,p}(\mathbf{R}^{n-1},E)$. Moreover, there exists a universal extension operator* ext *belonging to $\mathcal{L}\Big(B^{\sigma-1/p}_{p,p}(\mathbf{R}^{n-1},E), H^\sigma_p(\mathbf{R}^n,E)\Big)$ such that* $\mathrm{tr}\circ\mathrm{ext} = \mathrm{id}$ *on $B^{\sigma-1/p}_{p,p}(\mathbf{R}^{n-1},E)$.*

Remark 4.20 Again we refer to Amann [4] for an anisotropic version (see also Remark 4.11).

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