Wavelets in function spaces on cellular domains

Benjamin Scharf

TU Munich (Garching)

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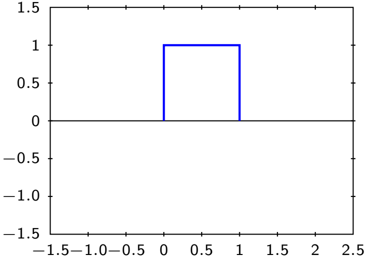
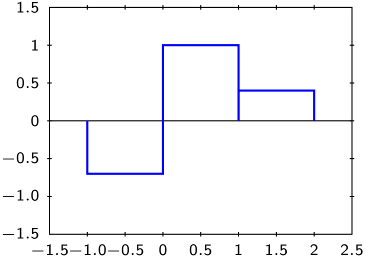
Table of contents

The Haar wavelet - the first/easiest wavelet (i)

The Haar wavelet (Alfred Haar 1910)



father Haar wavelet Φ_F

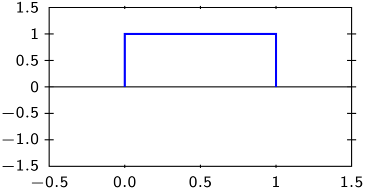


linear combinations of translated Φ_F's

Roughly speaking: Every function which is constant on intervals $[r, r+1]$, $r \in \mathbb{Z}$, can be written as a linear combination of the father wavelet.

The Haar wavelet - the first/easiest wavelet (ii)

Summing up



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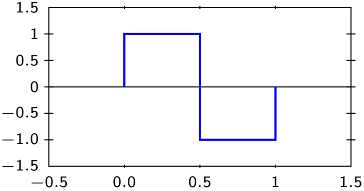
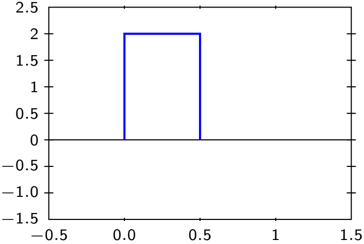


mother Haar wavelet Φ_M

yields



The Haar wavelet - the first/easiest wavelet (iii)

Define translated and dilated father and mother wavelets by

$$\Phi^j_{F,r}(x) := 2^{j/2}\Phi_F(2^j x - r) \text{ resp. } \Phi^j_{M,r}(x) := 2^{j/2}\Phi_M(2^j x - r).$$

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Then it holds

$$\Phi^1_{F,0} = \frac{\sqrt{2}}{2} \cdot \left(\Phi^0_{M,0} + \Phi^0_{F,0}\right)$$

or, more general,

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$$\Phi^1_{F,0} = \frac{\sqrt{2}}{2} \cdot \left(\Phi^0_{M,0} + \Phi^0_{F,0}\right)$$

or, more general,

$$\Phi^{j+1}_{F,0} = \frac{\sqrt{2}}{2} \left(\Phi^j_{M,0} + \Phi^j_{F,0}\right) \text{ for all } j \in \mathbb{N}_0.$$

Hence there is a transformation of linear combinations

$$f = \sum_r \underbrace{\lambda^J_{F,r}(f)}_{\text{coefficients}} \cdot \Phi^J_{F,r} \Leftrightarrow f = \sum_r \lambda^0_{F,r}(f) \cdot \Phi^0_{F,r} + \sum_{j \leq J-1} \sum_r \lambda^j_{M,r}(f) \cdot \Phi^j_{M,r}$$

Moreover $\left\{\Phi^0_{F,r}, \Phi^j_{M,r'}\right\}$ is an orthonormal system.

The Haar wavelet - the first/easiest wavelet (iv)

Theorem (classical Haar wavelet basis)

The set $\left\{\Phi^0_{F,r}, \Phi^j_{M,r'}\right\}^{j\in\mathbb{N}_0}_{r,r'\in\mathbb{Z}}$ *forms an orthonormal basis in* $L_2(\mathbb{R})$. *This means: An* $f \in L_1^{loc}(\mathbb{R})$ *belongs to the function space* $L_2(\mathbb{R})$ *if, and only if, there is a representation*

$$f = \sum_r \lambda^0_{F,r}(f)\cdot\Phi^0_{F,r} + \sum_{j\in\mathbb{N}_0}\sum_r \lambda^j_{M,r}(f)\cdot\Phi^j_{M,r}$$

with coefficients λ_F, λ_M *from the sequence space* $\ell_2(\mathbb{Z})$. *The representation is unique, linear and it holds*

$$\lambda^j_{M,r}(f) = (f, \Phi^j_{M,r}) \text{ resp. } \lambda^0_{F,r}(f) = (f, \Phi^0_{F,r})$$

[scalar product in $L_2(\mathbb{R})$*].*

The Haar wavelet - discussion

Advantages of the Haar wavelet basis:

- Orthonormal basis, i. e. unique representation, easy calculation of coefficients
- Computational aspect 1: Only 2 start functions (Φ_F and Φ_M) are necessary to remember
- Computational aspect 2: Local behaviour of the Haar wavelet (compact support)

$$\lambda^j_{M,r}(f) = (f, \Phi^j_{M,r}) = \int_{2^{-j}\cdot r}^{2^{-j}\cdot(r+1)} f(x) \cdot \Phi^j_{M,r}(x)\, dx.$$

Only need behaviour of f in a small region (rad. $\sim 2^{-j}$) around $2^{-j}r$

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Disadvantage of the Haar wavelet basis:

- Φ_F and Φ_M are not smooth, have jump discontinuities $\Rightarrow$ The Haar wavelet basis is able to describe edges, but not smooth behaviour. !!! Difference to Fourier transform/orthonormal basis (on torus) !!!

Smooth compact wavelet bases in dimension 1 (i)

The Main Goal

The more differentiable/smooth f, the faster the decay of the coefficients $\lambda^j_{M,r}(f)$ in $j \Rightarrow$ Better approximation/compression of smooth functions

Roughly speaking: $f \in C^k(\mathbb{R}) \Leftrightarrow \lambda^j_{M,r}(f) \lesssim j^{-k}$

This is not possible with Haar wavelet basis!

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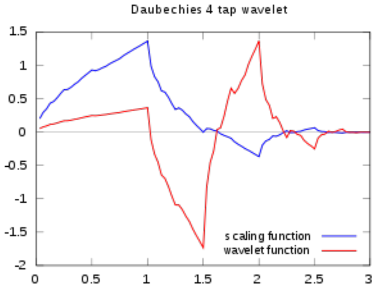
Requirements on compact wavelet basis in $C^u(\mathbb{R})$

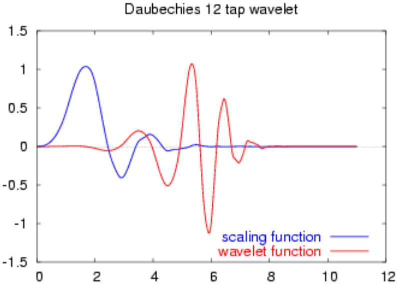
1. It holds $\Phi_M, \Phi_F \in C^u(\mathbb{R})$, $\Phi_M(x) = \Phi_F(x) = 0$ for $x \notin [0, c]$.
2. With $\Phi^j_{M,r}(x) := 2^{j/2}\Phi_M(2^j x - r)$ resp. $\Phi_{F,r}(x) := \Phi_F(x - r)$ the set $\left\{\Phi^0_{F,r}, \Phi^j_{M,r'}\right\}^{j\in\mathbb{N}_0}_{r,r'\in\mathbb{Z}}$ forms an orthonormal basis in $L_2(\mathbb{R})$.

Ingrid Daubechies 1988/book 1992: For every $u \in \mathbb{N}_0$ there are such functions Φ_M, Φ_F.

Smooth compact wavelet bases in dimension 1 (ii)

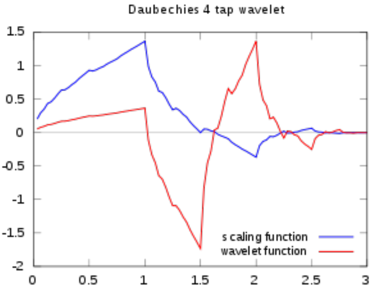


2 times differentiable Daubechies wavelets



6 times differentiable Daubechies wavelets

Smooth compact wavelet bases in dimension 1 (ii)



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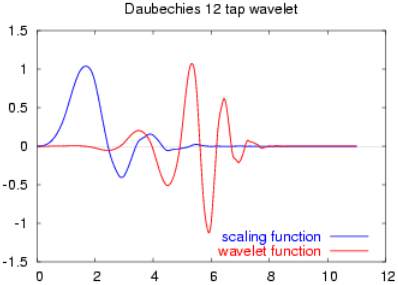

6 times differentiable Daubechies wavelets

The support grows with the smoothness, but remains compact.
A $C^\infty(\mathbb{R})$-wavelet with compact support is not possible.
Furthermore: Daubechies mother wavelets in $C^u(\mathbb{R})$ have u moment conditions, i. e. polynomials upto order $u-1$ are represented only by a linear combination of $\Phi^0_{F,r}$ and are orthogonal to $\Phi^j_{M,r}$.

Source: http://de.wikipedia.org/wiki/Daubechies-Wavelets

Wavelets on $\mathbb{R}^n$ - tensor product structure

One possible idea to generalize (Daubechies) wavelets to $\mathbb{R}^n$ is the tensor product structure: For this we define

$$\Phi^j_{G,r}(x) := 2^{jn/2} \prod_{k=1}^{n} \Phi^j_{G_k,r_k}(x) \text{ for } r \in \mathbb{Z}^n, j \in \mathbb{N}_0$$

with $r = (r_1, \dots, r_n)$ and $G = (G_1, \dots, G_n) \in \{F, M\}^n$.
($G = (F, \dots, F)$ is not allowed for $j \geq 1$).

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Theorem (Daubechies wavelet basis in $\mathbb{R}^n$)

Let Φ_M, Φ_F be Daubechies wavelet functions in $C^u(\mathbb{R})$. Then the system $\{\Phi^j_{G,r}\}^{j\in\mathbb{N}_0}_{r\in\mathbb{Z}^n} \subset C^u(\mathbb{R}^n)$ with G as above is an orthonormal basis in $L_2(\mathbb{R}^n)$.

Sobolev spaces $W_2^k(\mathbb{R}^n)$

Remember the Goal:

The smoother f is, the faster should be the decay of $\lambda^j_{M,r}(f)$ in j.

A way to describe integrability and smoothness are Sobolev spaces on $\mathbb{R}^n$:

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A way to describe integrability and smoothness are Sobolev spaces on $\mathbb{R}^n$:

Definition

Let $k \in \mathbb{N}$. Then

$$W_2^k(\mathbb{R}^n) := \{f \in L_2(\mathbb{R}^n) : D^\alpha f \in L_2(\mathbb{R}^n) \text{ for } |\alpha| \leq k\}.$$

The main result (here for $W_2^k(\mathbb{R}^n)$, in general for Besov- and Triebel-Lizorkin spaces) is the wavelet representation by Daubechies wavelets:

Wavelets for Sobolev spaces $W_2^k(\mathbb{R}^n)$

Theorem (Daubechies 89/92, Meyer 90/92, Triebel 06/08)

Let $k, u \in \mathbb{N}$, $u \geq k+1$ and $\Phi^j_{G,r} \in C^u(\mathbb{R}^n)$ be the functions of the n-dimensional Daubechies wavelet basis of $L_2(\mathbb{R}^n)$. Then $f \in L_2(\mathbb{R}^n)$ belongs to the Sobolev space $W_2^k(\mathbb{R}^n)$ iff

$$f = \sum_{j,r,G} \lambda^j_{G,r}(f) \cdot \Phi^j_{G,r}$$

and

$$\lambda \in w_2^k(\mathbb{Z}^n), \text{ i.e. } \sum_{j,r,G} \left(2^{jk} \cdot |\lambda^j_{G,r}(f)|\right)^2 < \infty.$$

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Table of contents

Function spaces on domains $\Omega \subset \mathbb{R}^n$

From a general point of view function spaces on domains $\Omega \subset \mathbb{R}^n$ can be introduced by restriction (quotient space), e. g. for Sobolev spaces on the unit cube:

Definition

Let Q be the unit cube in $\mathbb{R}^n$. Then

$$W_2^k(Q) := \{f \in L_2(Q) : f = g|Q \text{ for some } g \in W_2^k(Q)\},$$

$$\|f|W_2^k(Q)\| = \inf \|g|W_2^k(Q)\|$$

where the infimum is taken over all $g \in W_2^k(\mathbb{R}^n)$ with $g|Q = f$.

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where the infimum is taken over all $g \in W_2^k(\mathbb{R}^n)$ with $g|Q = f$.

In special cases there are equivalent (intrinsic) characterizations, here e. g.

$$f \in W_2^k(Q) \Leftrightarrow \sum_{|\alpha| \leq k} \|D^\alpha f|L_2(Q)\| < \infty \text{ (equivalent norms)}$$

Wavelets on domains $\Omega \subset \mathbb{R}^n$ (i)

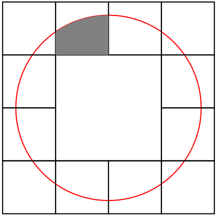
How to transfer wavelet bases from spaces of functions on $\mathbb{R}^n$ to spaces of functions on domains $\Omega \subset \mathbb{R}^n$?

Wavelets on domains $\Omega \subset \mathbb{R}^n$ (i)

How to transfer wavelet bases from spaces of functions on $\mathbb{R}^n$ to spaces of functions on domains $\Omega \subset \mathbb{R}^n$?

First idea: Take wavelets with compact support (Haar, Daubechies)!

Second idea: Take a function on Ω, extend it to $\mathbb{R}^n$, find a wavelet decomposition and restrict it to Ω



Problems: impractical (not intrinsic), basis property, orthogonality, moment conditions, (smoothness)

Solution: Construct wavelet bases (orthogonal or biorthogonal) directly on Ω - as orthogonal bases in $L_2(\Omega)$: Ciesielski, Figiel '83,'84, Triebel '06,'08

Wavelets for function spaces on domains Ω

Let $u \in \mathbb{N}_0$. Then

$$\Phi = \left\{ \Phi^j_\ell : j \in \mathbb{N}_0, \ell = 1, \ldots, N_j \right\} \subset C^u(\Omega)$$

is called a u-wavelet system in $\bar{\Omega}$ (adapted to $\mathbb{Z}^\Omega$) if it fulfils

- support conditions: Φ^j_ℓ has support in $\Omega \cap$ a cube of radius $\sim 2^{-j}$
- derivative conditions: Φ^j_ℓ belongs to $C^u(\Omega)$ and the derivatives are suitably bounded (in $\mathbb{R}^n$ this follows automatically)

Wavelets for function spaces on domains Ω

Let $u \in \mathbb{N}_0$. Then

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- support conditions: Φ_ℓ^j has support in $\Omega \cap$ a cube of radius $\sim 2^{-j}$
- derivative conditions: Φ_ℓ^j belongs to $C^u(\Omega)$ and the derivatives are suitably bounded (in $\mathbb{R}^n$ this follows automatically)

Additionally, the u-wavelet system is called oscillating if it fulfils

- (substitute) moment conditions: It holds

$$\left| \int_\Omega \psi(x) \Phi_\ell^j(x)\, dx \right| \lesssim 2^{-j\frac{n}{2} - ju} \left\| \psi | C^u(\Omega) \right\| \text{ for all } \psi \in C^u(\Omega)$$

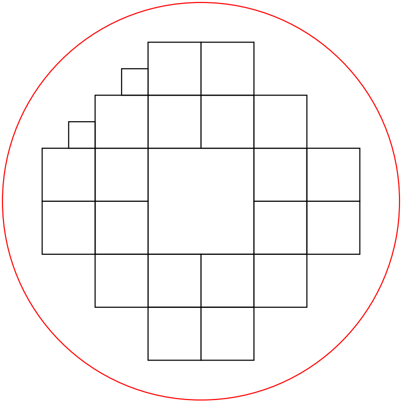
  if Φ_ℓ^j lies inside Ω or on the boundary (distance $\notin \left(c_1 2^{-j}, c_2 2^{-j}\right)$).

An oscillating u-wavelet system is called interior if it fulfils

- (further) interior support conditions: The support of Φ_ℓ^j has distance $\sim 2^{-j}$ to the boundary $\partial\Omega \Rightarrow$ all wavelets lie inside Ω away from $\partial\Omega$

Interior wavelets - for a ball

The support of the (first order) interior wavelets (Ω is a ball):



Interior wavelet bases in $L_2(\Omega)$ - the starting point

Theorem (Triebel 2008)

Let Ω be an arbitrary domain in $\mathbb{R}^n$. For any $u \in \mathbb{N}_0$ there is a

$$\Phi = \left\{ \Phi_\ell^j : j \in \mathbb{N}_0, \ell = 1, \ldots, N_j \right\} \subset C^u(\Omega)$$

which is

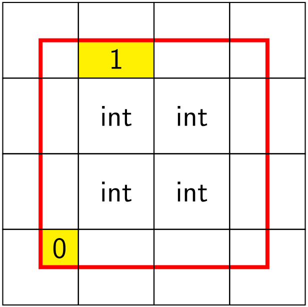
1. *an orthonormal basis in $L_2(\Omega)$,*
2. *an interior u-wavelet system*

simultaneously.

For $u = 0$ one can take the Haar wavelet suitably restricted to Ω.

Wavelet bases for $W_2^k(Q)$ - boundary wavelets

A u-wavelet system which should be a basis for $W_2^k(Q)$ cannot be interior for $k \geq 1$: Then elements of $W_2^k(Q)$ have boundary values on the faces of Q (traces), interior wavelets do not. We need boundary wavelets emerging from the boundary values of possibly every dimension $k = 0, \ldots, n-1$



Wavelets for $W_2^k(Q)$

Theorem (Triebel 2008 - Theorem 6.30 for spaces $F^s_{p,q}(Q)$)

Let

$$u, k \in \mathbb{N}_0, u > k \text{ and } k - \frac{m}{2} \notin \mathbb{N}_0$$

for $m = 1, \ldots, n$.
Then there is an oscillating u-wavelet system Φ which is a Riesz basis in the Sobolev space $W_2^k(Q)$,

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Then there is an oscillating u-wavelet system Φ *which is a Riesz basis in the Sobolev space* $W_2^k(Q)$*, i. e.: An element* $f \in L_2(Q)$ *belongs to* $W_2^k(Q)$ *iff it can be represented as*

$$f = \sum_{j \in \mathbb{N}_0} \sum_{r=1}^{N_j} \lambda_r^j(f) \cdot 2^{-\frac{jn}{2}} \Phi_r^j \qquad (1)$$

with λ *from the sequence space* $w_2^k(Q)$*. The representation* (1) *is unique, linear and*

$$\text{essentially } \lambda_r^j(f) \sim 2^{jn/2}(f, \Phi_r^j).$$

Traces on the boundary of cubes

To exclude the values $k - \frac{m}{2} \notin \mathbb{N}_0$ for $m = 1, \ldots, n$ is natural by the used method. The following proposition was the main part of the proof:

Proposition

Let

$$k \in \mathbb{N} \text{ and } k - \frac{m}{2} \notin \mathbb{N}_0 \text{ for } m = 1, \ldots, n.$$

Then it holds

$$\tilde{W}_2^k(Q) = \left\{ f \in W_2^k(Q) : \mathrm{tr}_\Gamma^{\bar{r}} = 0 \right\},$$

(all existing traces have to vanish!)

Here

$$\tilde{W}_2^k(Q) := \{ f \in W_2^k(\mathbb{R}^n) : \mathit{supp}\, f \subset \overline{Q} \}.$$

The spaces on the left have interior u-wavelet Riesz bases.

Table of contents

The situation for cubes Q in the exceptional cases

Roughly speaking:

- a C^∞-domain has only boundaries of dimension $n-1 \to$ exceptional values for $k - \frac{1}{2} \in \mathbb{N}_0$
- the cube Q has boundaries of dimension 0 to $n-1 \to$ exceptional values for $k - \frac{m}{2} \in \mathbb{N}_0$ for $m = 1, \ldots, n$

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Example (Grisvard '85, '92)

The space $W_2^1(Q) = F_{2,2}^1(Q)$ is exceptional: $k - \frac{2}{2} = 2 - 1$. Let $\Gamma = \partial\Omega = I_1 \cup I_2 \cup I_3 \cup I_4$. Then the trace space $\mathrm{tr}_\Gamma\, W_2^1(Q)$ is the collection of all tuples $g = (g_1, g_2, g_3, g_4)$ with

$$g_\ell \in H^{\frac{1}{2}}(I_\ell), \quad \ell = 1, 2, 3, 4$$

and

$$\int_0^{1/2} \frac{|g_1(t) - g_2(t)|^2}{t}\, dt < \infty, \text{ etc.}$$

Reinforced function spaces for cubes Q (i)

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Reinforced function spaces for cubes Q (i)

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Let

$$d(x) = \text{dist}(x, \partial\Omega) \text{ and } \Omega_\varepsilon := \{x \in \Omega : d(x) < \varepsilon\}.$$

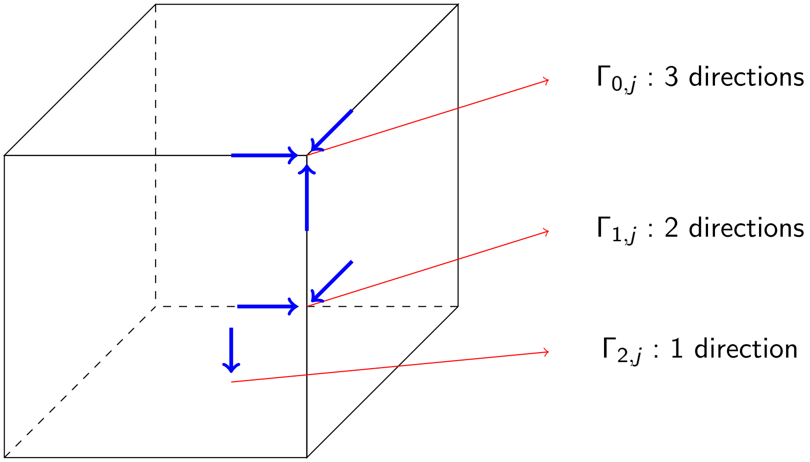
Let $\Gamma_{\ell,j}$ be the ℓ-dimensional faces of the cube Q. With $\mathbb{N}^n_{\ell,j}$ we denote the multi-indices with directions perpendicular to $\Gamma_{\ell,j}$.

Definition

We say that $f \in W^k_2(\mathbb{R}^n)$ has the reinforced property $R^{r,2}_\ell$ if, and only if,

$$d^{-\frac{n-\ell}{2}} \cdot D^\alpha f \in L_2((\mathbb{R}^n \setminus \Gamma_{\ell,j})_\varepsilon) \text{ for all } \alpha \in \mathbb{N}^n_{\ell,j}, |\alpha| = r \text{ and } j = 1, \dots, n_\ell.$$

Reinforced function spaces for cubes Q (ii)



Reinforced function spaces for cubes Q (iii)

Definition

Let $k \in \mathbb{N}$ and

$$W_2^{k,\text{rinf}}(Q) := W_2^{k,\text{rinf}}(\mathbb{R}^n \setminus \Gamma)_{|Q}$$

with inf-norm and

$$W_2^{k,\text{rinf}}(\mathbb{R}^n \setminus \Gamma) :=$$

$$\{f \in W_2^k(\mathbb{R}^n) : \forall\, 0 \leq \ell \leq n-1 : f \text{ fulfils } R_\ell^{r,2} \text{ if } r = k - \frac{n-\ell}{2} \in \mathbb{N}_0\}$$

Check for every dimension ℓ if the values are exceptional and if so, add reinforce property!

Wavelet bases for reinforced function spaces on cubes Q

Theorem (Scharf (2012) - for $F^s_{p,q}(Q)$-spaces)

Let $k \in \mathbb{N}$, $u > k$. Then there is an oscillating u-wavelet system which is a Riesz basis in $W_2^{k,\mathrm{rinf}}(Q)$ - i. e.

$$f \in W_2^{k,\mathrm{rinf}}(Q) \quad \Leftrightarrow \quad f = \sum_{j=0}^{\infty} \sum_{r=1}^{N_j} \lambda_r^j(f) \cdot 2^{-\frac{jn}{2}} \Phi_r^j$$

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- In the special case $k = 0$ ($L_p(\Omega)$) we can choose an interior u-wavelet system, for instance the Haar wavelet system - otherwise not.
- This theorem is a generalization of the wavelet theorem for $k - \frac{m}{2} \notin \mathbb{N}_0$ (Triebel 2008) since then there are no extra conditions.

An example - $W_2^{1,\mathrm{rinf}}(Q) = F_{2,2}^{1,\mathrm{rinf}}(Q)$ for $n = 2$ (i)

We have $k - \frac{2}{2} = r = 0$ - faces of dimension 0 are problematic. Hence

$$W_2^{1,\mathrm{rinf}}(Q) = F_{2,2}^{1,\mathrm{rinf}}(Q) = \left\{ f \in W_2^1(Q) : \int_Q |f(x)|^2 \frac{dx}{d(x)^2} < \infty \right\},$$

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Then by the theorem we find a (non-interior) oscillating u-wavelet system which is a Riesz basis. This is not possible (in this way) for $W_2^1(Q)$ - wavelet bases (at least using the definition here) for $W_2^1(Q)$ were not found yet.

Summary

What we managed to do:

- Constructed wavelet bases for suitable reinforced function spaces on the cube Q (polyhedron)
- Generalized the theorem of Triebel in [Tri08] for the Triebel-Lizorkin spaces $F^s_{p,q}(Q)$ (including the Sobolev spaces $W^k_p(Q)$, $H^s_p(Q)$) and eliminated the exceptional values

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What we did not manage to do:

- Show that such reinforcements are actually always necessary
- Extend this construction to a reasonable reinforced function space for general cellular domains (C^∞-domains, balls)
- Extend this construction to the Besov spaces $B^s_{p,q}(Q)$ - this should be possible, but uglier

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Thank you for your attention!