Equivalent norms and characterizations for Function Spaces

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The Spaces $S(\mathbb{R}^n)$ and $S'(\mathbb{R}^n)$

Let $\mathcal{S}(\mathbb{R}^n)$ be the set of the infinitely often differentiable functions $\varphi : \mathbb{R}^n \to \mathbb{C}$ for which the seminorms

$$\|\varphi\|_{K,L} := \sup_{x\in\mathbb{R}^n} (1+|x|^2)^{\frac{K}{2}} \sum_{|\alpha|\leq L} |D^\alpha \varphi(x)|$$

for $K, L \in \mathbb{N}_0$ are finite.

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for $K, L \in \mathbb{N}_0$ are finite.
We say that a linear map $f : \mathcal{S}(\mathbb{R}^n) \to \mathbb{C}$ is a tempered distribution if there exist a constant $c > 0$ and $K, L \in \mathbb{N}_0$ such that

$$|f(\varphi)| \leq c\|\varphi\|_{K,L}$$

for all $\varphi \in \mathcal{S}(\mathbb{R}^n)$. The set of all this linear mappings is denoted by $\mathcal{S}'(\mathbb{R}^n)$.
We define that f_j converges to f in $\mathcal{S}'(\mathbb{R}^n)$ iff $f_j(\varphi)$ converges to $f(\varphi)$ for all $\varphi \in \mathcal{S}(\mathbb{R}^n)$ (weak convergence).

Fourier transform of $S(\mathbb{R}^n)$ and $S'(\mathbb{R}^n)$

The Fourier transform $\hat{\varphi}$ of $\varphi \in \mathcal{S}(\mathbb{R}^n)$ is given by

$$\hat{\varphi}(\xi) := (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \varphi(x) e^{-ix\xi} \, dx.$$

We will denote the inverse Fourier transform by $\check{\varphi}$. It holds

$$\check{\varphi}(\xi) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \varphi(x) e^{ix\xi} \, dx.$$

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$$\hat{f}(\varphi) := f(\hat{\varphi}) \text{ for } \varphi \in \mathcal{S}(\mathbb{R}^n)$$

and analogously the inverse $\check{f}$. We have $\hat{f} \in S'(\mathbb{R}^n)$ iff $f \in S'(\mathbb{R}^n)$.

Regular distributions and convolution

If $f \in L_p(\mathbb{R}^n)$ for a $p \in [1, \infty]$ then we can think of f as an element of $\mathcal{S}'(\mathbb{R}^n)$ by defining

$$f(\varphi) := \int_{\mathbb{R}^n} f(x)\varphi(x)\, dx.$$

Distributions of such an integral form are called regular.

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For $f \in \mathcal{S}'(\mathbb{R}^n)$ and $\varphi \in \mathcal{S}(\mathbb{R}^n)$ we define the convolution of f and φ as the function

$$(f * \varphi)(x) := (2\pi)^{-\frac{n}{2}} \cdot f(\varphi(x - \cdot)) \text{ for all } x \in \mathbb{R}^n.$$

Then we have

$$(\varphi * f)\hat{} = \hat{\varphi} \cdot \hat{f}.$$

Resolution of unity

Let $\varphi_0 \subset \mathcal{S}(\mathbb{R}^n)$ such that *supp* $\varphi \subset \left\{|x| \leq \frac{3}{2}\right\}$ and $\varphi(x) = 1$ for $|x| \leq 1$.

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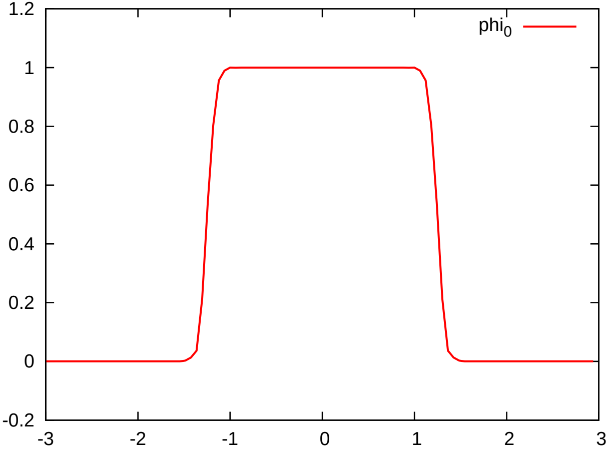

Resolution of unity (2)

We define

$$\varphi(x) := \varphi_0(x) - \varphi_0(2x) \text{ and } \varphi_j(x) := \varphi(2^{-j}x) \text{ for } j \in \mathbb{N}.$$

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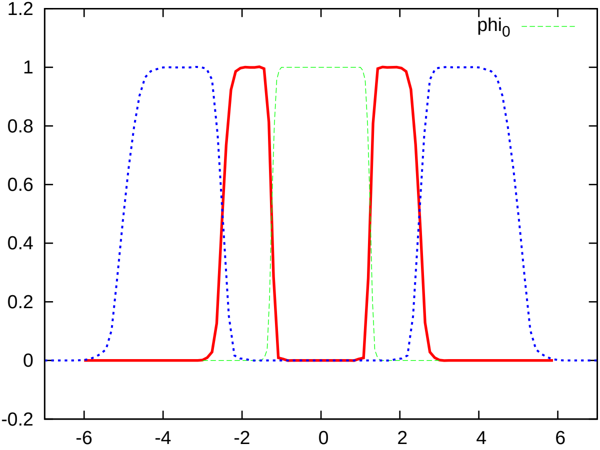

Resolution of unity (3)

Then we have

$$\begin{aligned} supp\ \varphi_j &\subset \left\{2^{j-1} \le |x| \le 2^{j+1}\right\} \\ |D^\alpha \varphi_j(x)| &\le c_\alpha 2^{-j|\alpha|}. \end{aligned} \tag{1}$$

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Furthermore we get

$$\sum_{j=0}^{N} \varphi_j(x) = \varphi_0(x) + \sum_{j=1}^{N} \left(\varphi_0(2^{-j}x) - \varphi_0(2^{-j+1}x)\right) = \varphi(2^{-N}x)$$

and therefore

$$\sum_{j=0}^{\infty} \varphi_j(x) = 1. \tag{2}$$

A sequence of functions $\{\varphi_j\}_{j=0}^{\infty}$ with $\varphi_j \in \mathcal{S}(\mathbb{R}^n)$ with the properties (1), (2) and a φ_0 as above will be called resolution of unity.

The definition of $B^s_{p,q}$

Let $\{\varphi_j\}_{j=0}^{\infty}$ be a resolution of unity. Let $0 < p \leq \infty$, $0 < q \leq \infty$ and $s \in \mathbb{R}$. For $f \in \mathcal{S}'(\mathbb{R}^n)$ we define

$$\|f|B^s_{p,q}\|^{\varphi} := \left(\sum_{j=0}^{\infty} 2^{jsq} \|(\varphi_j \hat{f})^{\vee}|L_p\|^q \right)^{\frac{1}{q}}$$

(modificated in case $q = \infty$) and

$$B^{s,\varphi}_{p,q} := \left\{ f \in \mathcal{S}'(\mathbb{R}^n) : \|f|B^s_{p,q}\|^{\varphi} < \infty \right\}.$$

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Then $(B^{s,\varphi}_{p,q}, \|\cdot|B^s_{p,q}\|^{\varphi})$ is a quasi-Banach space. It does not depend on the choice of the resolution of unity $\{\varphi_j\}_{j=0}^{\infty}$ in the sense of equivalent norms. So we denote it shortly by $B^s_{p,q}$.

The definition of $F^s_{p,q}$

Let $\{\varphi_j\}_{j=0}^{\infty}$ be a resolution of unity. Let $0 < p < \infty$, $0 < q \leq \infty$ and $s \in \mathbb{R}$. For $f \in \mathcal{S}'(\mathbb{R}^n)$ we define

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- Fourier multiplier theorems for functions whose Fourier transform have compact support.
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- the property $|D^\alpha \varphi_j(x)| \leq c_\alpha 2^{-j|\alpha|}$ which is clearly given a priori if we take $\varphi_j(x) := \varphi(2^{-j}x)$ for a $\varphi \in \mathcal{S}(\mathbb{R}^n)$,

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- the unit property $\sum_{j=0}^\infty \varphi_j(x) = 1$.

Question: Is there a possibility to weaken the conditions on φ_j?

First observations

For $f \in \mathcal{S}'(\mathbb{R}^n)$ we can write

$$(\varphi_j \hat{f})^\vee = (\varphi_j)^\vee * f.$$

So $\varphi_j \in \mathcal{S}(\mathbb{R}^n)$ is a natural (but not in any way necessary) assumption.

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If $\varphi \in \mathcal{S}(\mathbb{R}^n)$ has compact support, then $\check{\varphi} \in \mathcal{S}(\mathbb{R}^n)$ doesn't have (Paley Wiener theorem). Therefore: Even if f has compact support (in the distributional sense), $(\varphi_j \hat{f})^\vee$ in general not.

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So if we want this property, we need $\varphi \in \mathcal{S}(\mathbb{R}^n)$ such that $\check{\varphi} \in \mathcal{S}(\mathbb{R}^n)$ has compact support $\rightarrow$ the above used Fourier multiplier theorems won't work anymore.
From now on take $\varphi_j(x) := \varphi(2^{-j}x) \rightarrow$ only consider φ_0 and φ.

Triebel's approach

Hans Triebel: Theory of Function Spaces II, 1992.

- Assumes a priori that $f \in B^s_{p,q} \longrightarrow$ equivalent norms, but not characterizations (at first)
- Then$\varphi_0 \in C^\infty(\mathbb{R}^n)$ and $\varphi \in C^\infty(\mathbb{R}^n \setminus \{0\})$ need to fulfil (Tauberian) conditions on the decay at 0 and ∞ mainly in dependency on s (but also p and q), but don't need to be in $\mathcal{S}(\mathbb{R}^n) \leftarrow$ one has to care about defining a convolution

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- Not only discrete but also continuous norms

$$\left(\sum_{j=0}^{\infty} 2^{jsq} \|(\varphi(2^{-j}\cdot)\hat{f})^\vee | L_p\|^q \right)^{\frac{1}{q}}$$

$$\Rightarrow \|(\varphi_0 \hat{f})^\vee | L_p\| + \left(\int_0^1 t^{-sq} \|(\varphi(t\cdot)\hat{f})^\vee | L_p\|^q \, \frac{dt}{t} \right)^{\frac{1}{q}}$$

Rychkov's approach

V. S. Rychkov: On a Theorem of Bui, Paluszyński, and Taibleson. Proc. Steklov Inst. Math. 227, 280, 1999

Let $S \in \mathbb{Z}$ with $S \geq \lfloor s \rfloor$. Let φ_0, $\varphi \in \mathcal{S}(\mathbb{R}^n)$ and let there be an $\varepsilon > 0$ such that

$$|\varphi_0(x)| > 0 \text{ for } \{|x| < 2\varepsilon\}, \qquad |\varphi(x)| > 0 \text{ for } \left\{\frac{\varepsilon}{2} < |x| < 2\varepsilon\right\}$$

$$D^\alpha \varphi(0) = 0 \text{ for } |\alpha| \leq S.$$

Then it follows that

$$\|f|B^s_{p,q}\|_{\varphi_0,\varphi} := \|(\varphi_0 \hat{f})^\vee|L_p(E)\| + \left(\sum_{j=1}^{\infty} 2^{jsq} \|(\varphi(2^{-j}\cdot)\hat{f})^\vee|L_p\|^q\right)^{\frac{1}{q}}$$

(modificated in the case $q = \infty$) is an equivalent norm for $\|\cdot|B^s_{p,q}\|$. It holds

$$B^s_{p,q} = \left\{f \in \mathcal{S}'(\mathbb{R}^n) : \|f|B^s_{p,q}\|_{\varphi_0,\varphi} < \infty\right\}.$$

Comparison of the two theorems/proofs

Advantages of the Triebel proof

- More general result (see later part of the talk)
- No need of $\mathcal{S}(\mathbb{R}^n)$ functions
- Continuous norms (can be obtained through slightly modifications with Rychkov's proof, too)

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Advantages of the Rychkov proof

- Much easier proof
- Fewer conditions on φ, for $s < 0$ even no more conditions (this does not come out of Triebel's proof)
- Gives Characterizations for all s, p, q, not only equivalent norms

Local means

Let $k_0, k^0 \in \mathcal{S}(\mathbb{R}^n)$, $\hat{k_0}(0) \neq 0$, $\hat{k^0}(0) \neq 0$, $N \in \mathbb{N}_0$ with $2N > s$ and $k^N = \Delta^N k^0$. Then $\varphi_0 := \hat{k_0}$ und $\varphi := \widehat{k^N}$ fulfil the conditions of the theorem by Rychkov. Especially k_0 and k^0 can be chosen with support in $\{|x| \leq 1\}$.

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We have for a regular distribution f

$$(\hat{k^N}(2^{-j}\cdot)\hat{f})^{\vee}(x) = 2^{jn}\left(k^N(2^j\cdot) * f\right)(x) = \int k^N(y) f(x - 2^{-j}y)\, dy.$$

So for the computation of $(\hat{k^N}(2^{-j}\cdot)\hat{f})^{\vee}(x)$ you only need the information for f on the set $\left\{y \in \mathbb{R}^n : |x - 2^j y| \leq 1\right\}$.

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As a corollary we get:

If $f \in F^s_{p_0,q}$ has compact support, then $f \in F^s_{p,q}$ for all $0 < p \leq p_0$.

Differences

Let for simplicity (dim.) $n = 1$ and $\varphi(x) = e^{ix} - 1$. Then we have

$$(\varphi(t\cdot)\hat{f})^{\vee} = f(x+t) - f(x) =: \Delta_t f(x).$$

By iterating this we get for $\varphi(x) = \left(e^{ix} - 1\right)^M$

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Unfortunately $\varphi \notin \mathcal{S}(\mathbb{R}^n)$, but it fulfills

$$\varphi^{(m)}(0) = 0 \text{ for } m \leq M - 1.$$

It follows with Triebel's theorem : *If* $1 \leq p \leq \infty$, $1 \leq q \leq \infty$, $s > 0$ *and* $M > s$, *then*

$$\|f|L_p\| + \left(\int_0^1 t^{-sq} \|\Delta_t^M f(\cdot)|L_p\|^q \, \frac{dt}{t}\right)^{\frac{1}{q}}$$

is an equivalent norm for $\|\cdot|B^s_{p,q}(\mathbb{R})\|$.

Further applications

Corollary:
Let $1 \le p \le \infty$, $1 \le q \le \infty$, $0 < s < 1$ *and* $f \in B^s_{p,q}(\mathbb{R})$. *Then* $|f| \in B^s_{p,q}$. Follows from

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Further applications?

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Further applications?

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- Pointwise multipliers
- Diffeomorphisms
- Classical Besov Spaces and intrinsic characterizations
- Atomic and subatomic decompositions

The end

Thank you for your attention

Questions (if time)?