How to steer high-dimensional Cucker-Smale systems to consensus using low-dimensional information only

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Overview

The Cucker-Smale model - Introdruction

... a dynamical system used to describe the nature of a group of moving agents, i.e. birds, formation/evolution of languages etc.

$$\begin{cases} \dot{x}_i(t) = v_i(t) \\ \dot{v}_i(t) = \frac{1}{N} \sum_{j=1}^{N} a(\|x_j(t) - x_i(t)\|) \cdot (v_j(t) - v_i(t)), \end{cases}$$

where $x_1, \ldots, x_N, v_1, \ldots, v_N \in \mathbb{R}^d$ with given initial values at time 0 and a is a *non-increasing pos. Lipschitz function*.

Cucker and Smale: $$a(x) = \frac{K}{(\sigma^2 + x^2)^\beta}, \; K, \sigma > 0, \beta \geq 0$$

Ref.: *F. Cucker and S. Smale. Emergent behavior in flocks. IEEE Trans. Automat. Control 52(5):852–862, 2007.*
F. Cucker and S. Smale. On the mathematics of emergence. Jpn. J. Math. 2(1):197–227, 2007.

The Cucker-Smale model - Main parameters

To measure the distances of the particles as well as their velocities we introduce:

$$X(t) := \frac{1}{2N^2} \sum_{i,j=1}^{N} \|x_i(t) - x_j(t)\|^2 = \frac{1}{N} \sum_{i=1}^{N} \|x_i(t) - \overline{x}(t)\|^2 = \overline{x^2} - \overline{x}^2,$$

$$V(t) := \frac{1}{2N^2} \sum_{i,j=1}^{N} \|v_i(t) - v_j(t)\|^2 = \frac{1}{N} \sum_{i=1}^{N} \|v_i(t) - \overline{v}(t)\|^2 = \overline{v^2} - \overline{v}^2$$

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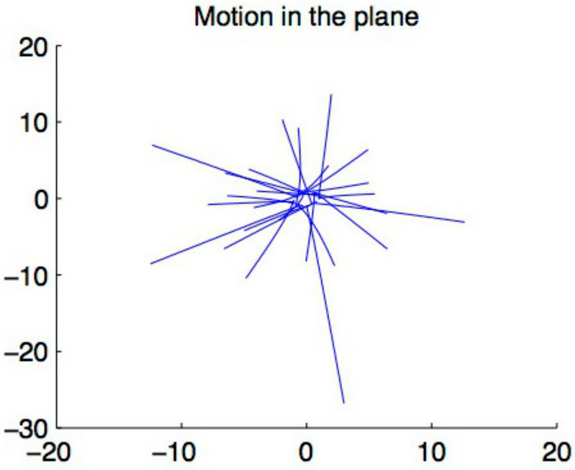
The main question is: Does the system tend to consensus?

$$?\ \lim_{t\to\infty} v_i(t) = \overline{v} \text{ or equivalently } \lim_{t\to\infty} V(t) = 0\,?$$

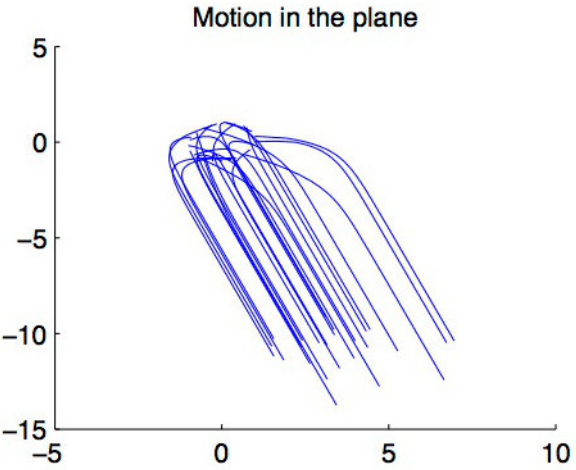
This would imply: The system moves as a swarm, i.e.

$$x(t) \approx x(t_0) + (t - t_0)\overline{v}$$

The Cucker-Smale model - Explosion



The Cucker-Smale model - Consensus



The Cucker-Smale model - Consensus

Lemma (Lyapunov functional behaviour)

It holds

$$\frac{d}{dt}V(t) \leq a\left(\sqrt{2NX(t)}\right)\sqrt{V(t)} \text{ as long as V(t)} > 0.$$

Hence: If $X(t)$ is bounded, the system tends to consensus.

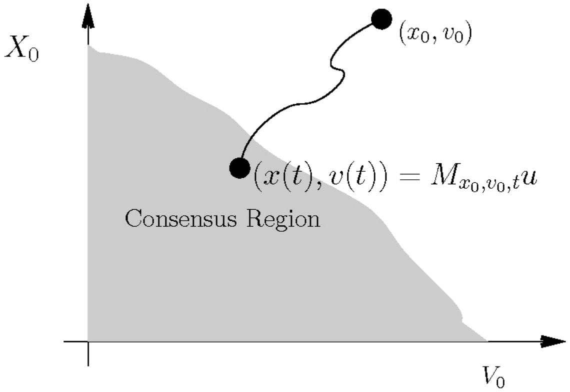
Theorem (Ha, Ha, Kim 2010)

$$\text{If } \int_{\sqrt{X(0)}}^{\infty} a\left(\sqrt{2N}r\right)\, dr \geq \sqrt{V(0)}, \text{ then } \lim_{t\to\infty} V(t) = 0.$$

Ref.: *S.-Y. Ha, T. Ha, and J.-H. Kim. Emergent behavior of a Cucker-Smale type particle model with nonlinear velocity couplings. IEEE Trans. Automat. Control 55(7):1679–1683, 2010.*

The consensus manifold

Idea: If we are not in the consensus manifold, use (sparse) control



Ref.: *M. Caponigro, M. Fornasier, B. Piccoli, and E. Trelat. Sparse stabilization and control of the Cucker-Smale model. Math. Contr. Relat. Field. 3(4):447–466, 2013.*

Sparsely Controlled Cucker-Smale system

Goal: Stear the system (using sparse control) to the consensus manifold and then stop the control.

Minimize the time to consensus and the number of agents to act on:

$$\begin{cases} \dot{x}_i(t) = v_i(t) \\ \dot{v}_i(t) = \frac{1}{N}\sum_{j=1}^{N} a(\|x_j(t) - x_i(t)\|) \cdot (v_j(t) - v_i(t)) + u_i(t) \end{cases}$$

using $\ell_1^N - \ell_2^d$-norm constraint (compare to compressed sensing)

$$\sum_{i=1}^{N} \|u_i(t)\|_2 \leq \Theta.$$

Observe: The control only acts on the most stubborn guy! (shepherd dog/Schäferhund strategy)

How to construct controls - Sample and Hold

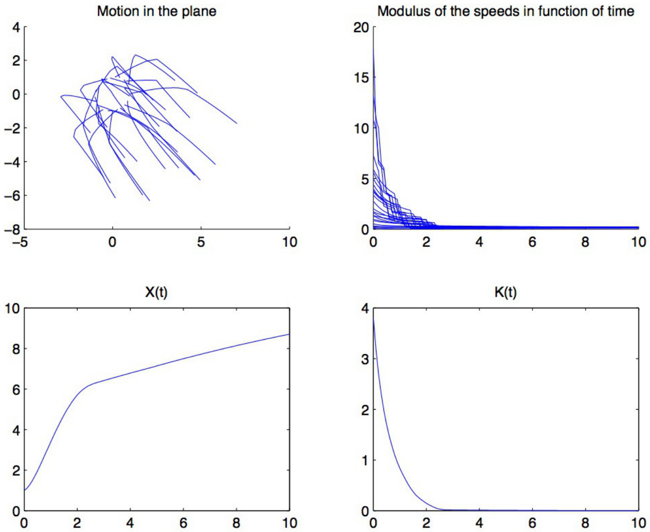
Sample and Hold idea: First construct solutions with controls constant on intervals $[k\tau, (k+1)\tau]$ - time-sparse controls. The ℓ_1^N-optimization leads to the sparse control

$$u_i = \begin{cases} -\Theta \frac{v_{\hat{\imath}}^{\perp}}{\|v_{\hat{\imath}}^{\perp}\|} & \text{if } \hat{\imath} \text{ is first } i : \|v_i^{\perp}(k\tau)\| = \max_{j=1,\dots,N} \|v_j^{\perp}(k\tau)\| \\ 0 & \text{otherwise} \end{cases}$$

Theorem (Caponigro, Fornasier, Piccoli, Trelat 2012)

For every Θ (constraint size) there exists $\tau_0 > 0$ such that for all sampling times $\tau \in (0, \tau_0]$ the sampling solution of the controlled Cucker-Smale system reaches the consensus region in finite time.

An example of sparse control



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Introduction

The case $N \to \infty$ (large number of agents) is widely considered in the literature:

- locations, velocities $\Rightarrow$ density distributions
- dynamical system, ODE $\Rightarrow$ PDE

Ref.: *M. Fornasier and F. Solombrino. Mean-field optimal control, to appear in ESAIM, Control Optim. Calc. Var., arXiv:1306.5913, 2013.*

Introduction

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The case $d \to \infty$ (high dimension/many coordinates/variables) is of our interest.

Example: Social movement (panic), Financial movement

Here: x not locations, rather variables/state of the system
(Health, pulse, strength; situation of the market, IFO-Index etc.)
v describes the movement towards consensus

$\Rightarrow$ Goal: Panic prevention, Black Swan prevention

Reduction of the dimension of Cucker-Smale

Idea: Consider reduction of the system by $M \in \mathbb{R}^{k \times d}$ (first no control)

$$\frac{\partial}{\partial t} Mv_i(t) = \frac{1}{N} \sum_{j=1}^{N} a(\|x_j(t) - x_i(t)\|) \cdot (Mv_j(t) - Mv_i(t))$$

$$\overset{\text{wish}}{\sim} \frac{1}{N} \sum_{j=1}^{N} a(\|Mx_j(t) - Mx_i(t)\|) \cdot (Mv_j(t) - Mv_i(t))\,.$$

Idea: Consider low-dimensional Cucker-Smale system in $\mathbb{R}^k$ with $(y_0, w_0) = (Mx_0, Mv_0)$ as initial values.

Ref.: *M. Fornasier, J. Haskovec and J. Vybiral. Particle systems and kinetic equations modeling interacting agents in high dimension. SIAM J. Multiscale Mod. Sim. 9(4):1727-1764,2011*

Johnson-Lindenstrauss matrices

Lemma (Johnson-Lindenstrauss matrices (JLM))

Let $x_1, \ldots, x_N$ be points in $\mathbb{R}^d$. Given $\varepsilon > 0$, there exists a constant

$$k_0 = \mathcal{O}(\varepsilon^{-2} \log N), \tag{1}$$

such that for all integers $k \geq k_0$ there is a $k \times d$ matrix M for which

$$(1 - \varepsilon)\|x_i\|^2 \leq \|Mx_i\|^2 \leq (1 + \varepsilon)\|x_i\|^2, \text{ for all } i = 1, \ldots, N.$$

How to construct a Johnson-Lindenstrauss matrix for N points and ε?

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How to construct a Johnson-Lindenstrauss matrix for N points and ε?

Only stochastic constructions are known (for k_0 as in (1))!

- entries $a_{ij} = \pm 1$ (Bernoulli) or $\mathcal{N}(0,1)$
- partial Fourier matrix (k of d random rows, mult. columns by ± 1)
- ...

These (correctly normalized) matrices work for given points with high probability. One does not need to know x_i!

A continuous Johnson-Lindenstrauss Lemma

Need the Johnson-Lindenstrauss property for continuous trajectories!

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Lemma (Bongini, Fornasier, Scharf 2014 (BFS))

Let $\varphi : [0,1] \to \mathbb{R}^d$ *be a Lipschitz function (bound* L_φ*),* $0 < \varepsilon' < 1/2$*,* $\delta > 0$ *and* $M \in \mathbb{R}^{k \times d}$ *be a Johnson-Lindenstrauss matrix for*

$$\varepsilon = \frac{\varepsilon'}{2} \quad \text{and} \quad N \geq 4 \cdot L_\varphi \cdot \frac{\sqrt{d}+2}{\delta \varepsilon'}$$

points with high probability. Then for every $t \in [0,1]$ *one of the following holds (with the same high probability):*

$$(1-\varepsilon')\|\varphi(t)\| \leq \|M\varphi(t)\| \leq (1+\varepsilon')\|\varphi(t)\|$$

$$\text{or}$$

$$\|\varphi(t)\| \leq \delta \text{ and } \|M\varphi(t)\| \leq \delta.$$

Remark: Cucker-Smale traj. need not to have bounded curvature.

Dimension reduction without control

Theorem (BFS 2014)

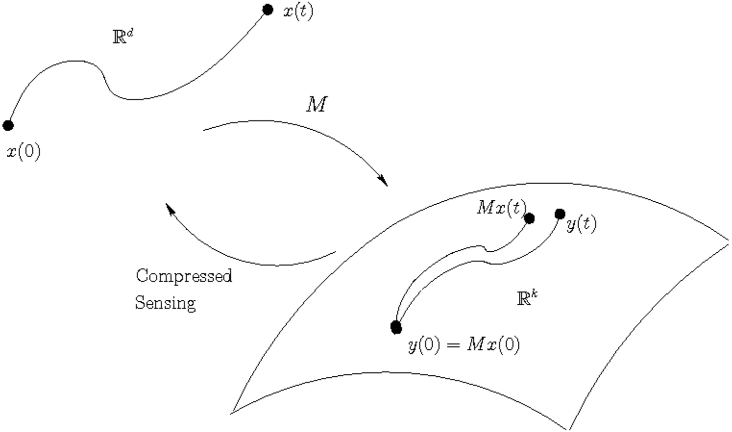
- $(x, v) = (x_1, \dots, x_N, v_1, \dots, v_N)$: *solution of the d-dimensional Cucker-Smale system for given initial values* $x(0), v(0) \in \mathbb{R}^{N\times d}$,
- $M \in \mathbb{R}^{k\times d}$ *continuous Johnson-Lindenstrauss matrix for* $\|x_i(t) - x_j(t)\|$ *for given* $\varepsilon, \delta > 0$ *and all* $t \in [0, T]$

Then the k(low)-dimensional Cucker-Smale system (y, w) *with initial values* $(y(0), w(0)) = (Mx(0), Mv(0))$ *stays close to the projected d(high)-dimensional Cucker-Smale system, i.e.,*

$$\|y(t) - Mx(t)\| + \|w(t) - Mv(t)\| \lesssim (\varepsilon + \delta)e^{Ct}, \quad t \leq T.$$

Remark: Fornasier, Haskovec, and Vybiral showed an analog for a general class of Cucker-Smale-like systems with bounded curvature.

Low-dim. and projection of high-dim. stay close



The plan – dimension reduction with control (i)

Consider the high-dimensional system, given $(x(0), v(0))$,

$$\begin{cases} \dot{x}_i(t) = v_i(t) \\ \dot{v}_i(t) = \frac{1}{N} \sum_{j=1}^{N} a(\|x_j(t) - x_i(t)\|) \cdot (v_j(t) - v_i(t)) + u_i^h(t) \end{cases}$$

and the low-dimensional system, $(y(0), w(0)) = (Mx(0), Mv(0))$,

$$\begin{cases} \dot{y}_i(t) = y_i(t) \\ \dot{w}_i(t) = \frac{1}{N} \sum_{j=1}^{N} a(\|y_j(t) - y_i(t)\|) \cdot (w_j(t) - w_i(t)) + u_i^\ell(t) \end{cases}$$

with analog constraints as before.

We want to act sparse and we want to compute the controlled agent only with low-dimensional information!

The plan – dimension reduction with control (ii)

1. project the high-dimensional initial values with JLM M to low-dimension, $(y(0), w(0)) = (Mx(0), Mv(0))$
2. choose the index of sparse control (one agent) from the low-dimensional system, apply it to both systems!

$$u_i^\ell = \begin{cases} -\Theta \frac{w_{\hat{\imath}}^\perp}{\|w_{\hat{\imath}}^\perp\|} & \text{if } \hat{\imath} \text{ is first } i : \|w_i^\perp(k\tau)\| = \max_{j=1,\dots,N} \|w_j^\perp(k\tau)\| \\ 0 & \text{otherwise} \end{cases}$$

$$u_i^h = \begin{cases} -\Theta \frac{v_{\hat{\imath}}^\perp}{\|v_{\hat{\imath}}^\perp\|} & \text{if } i = \hat{\imath} \\ 0 & \text{otherwise} \end{cases}$$

3. show: if M is a suitable (continuous) JLM, then
   - the systems stay close to each other (as before in the projected sense) and
   - the control is reasonable for both systems (decays of $W(t)$ and $V(t)$ are fast enough).

The main result

Theorem (BFS 2014)

Let $M \in \mathbb{R}^{k \times d}$ and $\Theta > 0$ (arbitrary constraint)

- (x, v) *and* (y, w): *solutions of the d-dimensional and k-dimensional (so) controlled Cucker-Smale systems, initial values* $(x(0), v(0))$ *and* $(Mx(0), Mv(0))$, *resp. ,*
- *assume* M *is continuous Johnson-Lindenstrauss matrix for* $\|x_i(t) - x_j(t)\|$ *and* $\|v_i(t) - v_j(t)\|$ *sufficiently long,* ε *and* δ *sufficiently (very) small (depending on the initial values and* Θ*)*

Then for every $\tau \in (0, \tau_0]$ *both so controlled Cucker-Smale systems*

1. *stay close to each other (in the projected sense),*
2. *reach consensus manifold in finite time, and*
3. *have reached consensus manifold, when a certain parameter of the low-dimensional systems falls below a known threshold.*

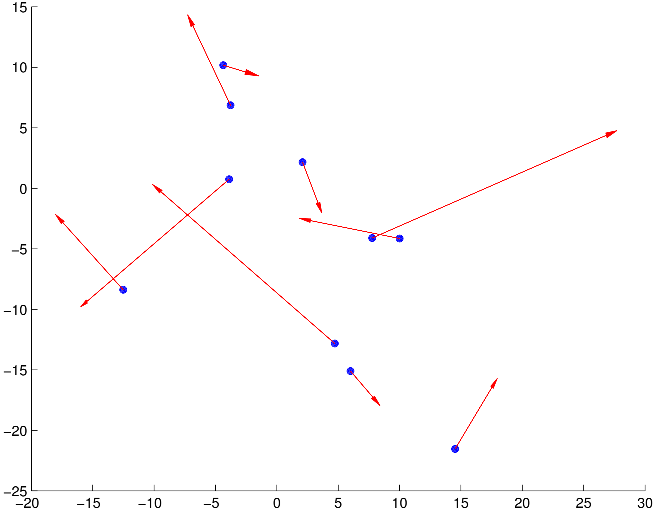
Remarks and open problems

- One can compute the index of the agent to control from low dimension, but the direction of its control needs to be observed from high dimension.
- ε and δ (and everything else) do not depend on d, but heavily on N
- If $\Theta >> 0$ and is constant with N, then ε has to be chosen s.t.

$$\varepsilon \sim N^{-\mu_1} \cdot e^{-\mu_2 N}, \quad \mu_1, \mu_2 > 0.$$

- In the theorem we suppose M is a JLM for the trajectories, but: the initial values of the low-dimensional system and hence the controls depend on M
  $\Rightarrow$ as a worst case scenario we take all possible trajectories into account (exponentially in the number of control switches).

Numerics: random $N = 10, d = 500, \theta = 20, \beta = 0.65$



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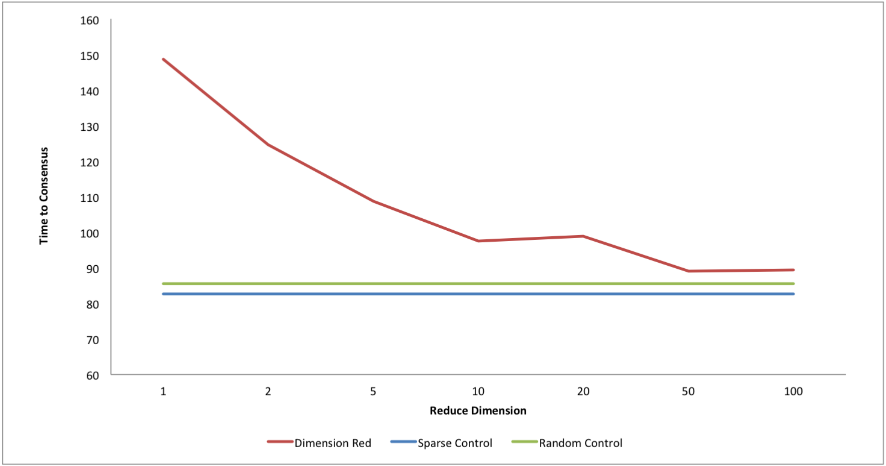
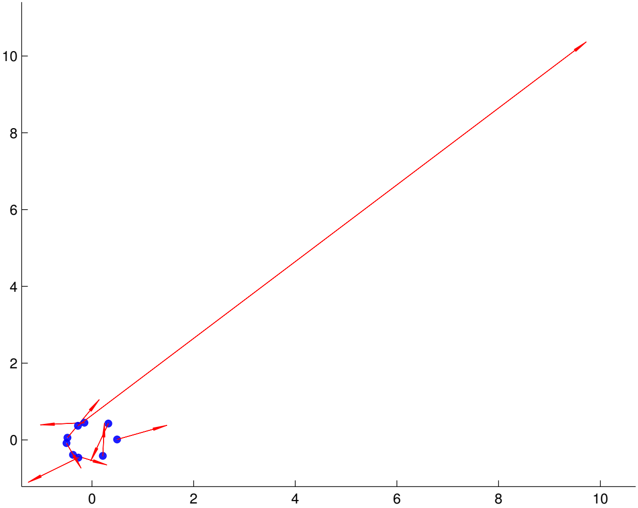
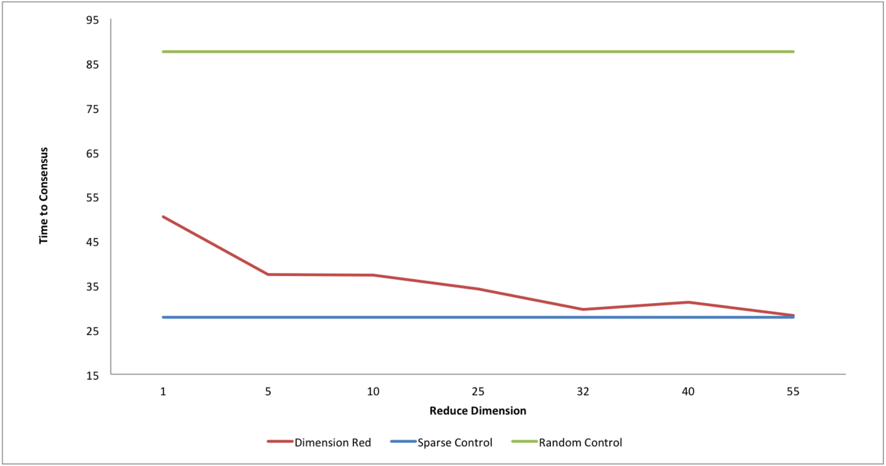


Numerics: outlier $N = 9, d = 100, \theta = 5, \beta = 0.6$ (i)



Numerics: outlier $N = 9, d = 100, \theta = 5, \beta = 0.6$ (ii)



The End

Room for improvement

Thank you for your attention

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