Besov regularity of solutions of the p-Laplace equation

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Overview

The p-Laplace - Introduction

$\Omega \subset \mathbb{R}^d$ Lipschitz domain, d dimension, $1 < p < \infty$

Inhomogeneous p-Laplace equation:

$$\Delta_p u := \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = f \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega.$$

Variational (weak) formulation:

$$\int_\Omega \left\langle |\nabla u|^{p-2}\,\nabla u, \nabla v \right\rangle \mathrm{d}x = \int_\Omega f\, v \,\mathrm{d}x \quad \text{for all} \quad v \in C_0^\infty(\Omega)$$

- has a unique solution $u \in \mathring{W}_p^1(\Omega)$ for $f \in W_{p'}^{-1}(\Omega)$,
- has model character for nonlinear problems, similar to the Laplace equation ($p = 2$) for linear problems

nice and free introduction: P. Lindqvist. Notes on the p-Laplace equation, 2006.

http://www.math.ntnu.no/~lqvist/p-laplace.pdf

Sobolev and Besov spaces

$W_p^s(\Omega)$: Sobolev space of smoothness s and integrability p on Ω

$B_{p,p}^s(\Omega)$: Besov space of smoothness s and integrability p on Ω

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Wavelet representation: $\eta_{I,p} = |I|^{1/2-1/p}\,\eta_I$ p-normalized wavelets

$$g \in B^s_{p,p}(\mathbb{R}^d) \Leftrightarrow g = P_0(g) + \sum_I \sum_{\eta\in\Psi} \langle g, \eta_{I,p'}\rangle \eta_{I,p}$$

$$\text{and} \quad \left\| P_0(g) \,\middle|\, L_p(\mathbb{R}^d) \right\| + \left\| \langle g, \eta_{I,p'}\rangle \,\middle|\, b^s_{p,p}(\mathbb{R}^d) \right\| < \infty$$

Here

$$\left\| \langle g, \eta_{I,p'}\rangle \,\middle|\, b^s_{p,p}(\mathbb{R}^d) \right\|^p = \sum_I \sum_{\eta\in\Psi} |I|^{-sp/d} \left|\langle g, \eta_{I,p'}\rangle\right|^p$$

more smoothness $\Leftrightarrow$ more decay of the wavelet coefficients

Trivial embedding: $B^{s+\varepsilon}_{p,p}(\Omega) \hookrightarrow W^s_p(\Omega) \hookrightarrow B^s_{p,p}(\Omega)$

Linear and Adaptive approximation by wavelets (i)

How to approximate $f \in B^s_{p,p}(\Omega)$, Ω bounded, by wavelet basis?

Linear approximation f_k of f (order k: $\sim 2^{kd}$ terms):

$$f_k = P_0(g) + \sum_{|I| \geq 2^{-k}} \sum_{\eta \in \Psi} \left\langle g, \eta_{I,p'} \right\rangle \eta_{I,p}$$

It holds

$$f \in B^s_{p,p}(\Omega) \text{ (or } W^s_p(\Omega)) \Rightarrow \|f - f_k \mid L_p(\Omega)\| \lesssim 2^{-ks}.$$

Linear and Adaptive approximation by wavelets (ii)

Adaptive approximation f_k of f (order k: $\sim 2^{kd}$ terms):

$$f_k^D = P_0(g) + \sum_{(I,\eta)\in D} \langle g, \eta_{I,p'} \rangle \eta_{I,p} \text{ with } |D| = 2^{kd}$$

best m-term approximation: choose D to minimize

$$\left\| f - f_k^D \,\middle|\, L_p(\Omega) \right\| : \text{ take } 2^{kd} \text{ largest wavelet coefficients!}$$

Let $\frac{1}{\tau} = \frac{\sigma}{d} + \frac{1}{p}$, in particular $\tau < 1$ possible. It holds

$$f \in B^{\sigma}_{\tau,\tau}(\Omega) \Rightarrow \| f - f_k \,|\, L_p(\Omega) \| \sim 2^{-k\sigma}$$

Besov regularity is the maximal possible convergence rate of an adaptive algorithm $\Rightarrow$ how much higher than Sobolev regularity?

Linear and Adaptive approximation by wavelets (iii)

The main reason is the following computation:

Theorem

Let $\frac{1}{\tau} = \frac{\sigma}{d} + \frac{1}{p}$, $x \in \ell_\tau$ and x^* its non-increasing rearrangement. Then

$$\|x^* - x_k^*\|_p \leq k^{-\frac{\sigma}{d}} \|x\|_\tau,$$

where x_k^* is the cut-off of x^* after the k first terms.

Proof:

Assume w.l.o.g. that $\|x\|_\tau = 1$. Then

$$|x^*(j)|^\tau \leq |x^*(k)|^\tau \leq \frac{1}{k}\|x^*\|_\tau^\tau = \frac{1}{k} \cdot \quad \text{for} \quad j > k.$$

Therefore

$$\|x^* - x_k^*\|_p^p \leq \|x^* - x_k^*\|_\infty^{p-\tau} \cdot \|x^* - x_k^*\|_\tau^\tau \leq k^{\frac{\tau - p}{\tau}} \cdot 1 = k^{-\frac{\sigma}{d}p}.$$

Sobolev regularity for $p = 2$, the linear case

Theorem (Jerison, Kenig 1981,1995, Theorem B)

Positive: Lipschitz domain $\Omega \in \mathbb{R}^d$, $f \in L_2(\Omega)$. *Then the solution u of*

$$\Delta u = f \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega$$

belongs to $W_2^{3/2}(\Omega)$.

Negative: For any $s > 3/2$ *there exists a Lipschitz domain* Ω *and smooth* f *s.t.* u *with*

$$\Delta u = f \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega$$

does not belong to $W_2^s(\Omega)$.

Careful! $\exists\ \mathcal{C}^1$-domain Ω and $f \in W_2^{-1/2}(\Omega)$ such that $u \notin W_2^{3/2}(\Omega)$

D. Jerison, C.E. Kenig. The Inhomogeneous Dirichlet Problem in Lipschitz Domains. J. Funct. Anal. 130, 161–219, 1995.

Besov regularity for $p = 2$ (i)

Theorem (Dahlke,DeVore '97; Jerison,Kenig '95; Hansen 2013)

Lipschitz domain $\Omega \in \mathbb{R}^d$, $f \in W_2^{\gamma}(\Omega)$ *for* $\gamma \geq \max\left(\frac{4-d}{2d-2}, 0\right)$. *Then the solution* u *of*

$$\Delta u = f \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega$$

belongs to $B^{\sigma}_{\tau,\tau}(\Omega)$, $\frac{1}{\tau} = \frac{\sigma}{d} + \frac{1}{p}$, *for any* $\sigma < \frac{3}{2} \cdot \frac{d}{d-1}$.

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- Besov reg. always better than $3/2$, the maximal Sobolev regularity
- proof by a general embedding:
  small global Sobolev regularity + better local (weighted) Sobolev regularity (Babuska-Kondratiev) result in better Besov regularity!

S. Dahlke, R.A. DeVore. Besov regularity for elliptic boundary value problems. Comm. Partial Differential Equations, 22(1–2), 1–16, 1997.

M. Hansen, n-term approximation rates and Besov regularity for elliptic PDEs on polyhedral domains, to appear in J. Found. Comp. Math.

Besov regularity for $p = 2$ (ii)

Proof Idea:

- extend u to $\mathbb{R}^n$ and take its wavelet decomposition – 3 parts
  1. father wavelets (independent of regularity)
  2. interior and exterior wavelets $\eta_{I,p}$ with

$$\operatorname{dist}(I, \partial\Omega) \gtrsim \operatorname{diam}(I) \tag{1}$$

  3. boundary wavelets $\eta_{I,p}$; (1) doesn't hold

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- handle 3 parts separately
  1. no problem
  2. use weighted Sobolev reg.: If $f \in L_2(\Omega)$, then solution $u \in W_2^2(\Omega, w)$, weigth w exploding at the boundary (Babuska-Kondratiev spaces)
  3. use global Sobolev reg.: If $f \in L_2(\Omega)$, then solution $u \in W_2^{3/2}(\Omega)$, use counting argument:

$$\#\{\eta_{I,p} \text{ boundary wav.}, \mathrm{diam}(I) \sim 2^{-j}\} \sim 2^{j(d-1)}$$

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Sobolev regularity of the p-Laplace

Theorem (Ebmeyer 2001, 2002, Savare 1998)

$\Omega \subset \mathbb{R}^d$ bounded polyhedral domain, $d \geq 2$, $1 < p < \infty$, $f \in L_{p'}(\Omega)$. If $\Delta_p u = f$ and $u = 0$ on $\partial\Omega$, then

$$V := |\nabla u|^{\frac{p-2}{2}} \nabla u \in W_2^{1/2-\varepsilon}(\Omega) \text{ for all } \varepsilon > 0 \qquad (2)$$

Furthermore

$$|\nabla u| \in L_q(\Omega) \text{ for } q < \frac{pd}{d-1}$$

and

$$u \in \begin{cases} W_{\widetilde{p}}^{3/2-\varepsilon}(\Omega), & \text{if } 1 < p \leq 2, \\ W_p^{1+1/p-\varepsilon}(\Omega), & \text{if } p \geq 2, \end{cases} \qquad \widetilde{p} = \frac{p}{1 - \frac{2-p}{2d}} > p.$$

Open question: Does (2) hold for general Lipschitz domains?

C. Ebmeyer. Nonlinear elliptic problems with p-structure under mixed boundary value conditions in polyhedral domains. Adv. Diff. Equ., 6:873–895, 2001.

Local Hölder regularity of the homogen. p-Laplace

Replacement for the local (weighted) Sobolev regularity ($p = 2$)

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Replacement for the local (weighted) Sobolev regularity ($p = 2$)

Theorem (Lewis 1983; Ural'ceva; Evans; DiBenedetto;...)

$\Omega \subset \mathbb{R}^d$ *bounded open set,* $d \geq 2$, $1 < p < \infty$. *There exists* $\alpha \in (0,1]$ *s.t.* u *with* $\Delta_p u = 0$ *fulfils:* $\forall\, x_0 \in \Omega, r > 0$ *s.t.* $B(x_0, 64r) \subset \Omega$

$$\max_{x \in B(x_0,r)} |\nabla u(x)| \leq C \left(\fint_{B(x_0,32r)} |\nabla u|^p dx \right)^{1/p} \leq C \cdot r^{-d/p},$$

$$\max_{x,y \in B(x_0,r)} |\nabla u(x) - \nabla u(y)| \leq C \cdot r^{-\alpha} \left(\fint_{B(x_0,32r)} |\nabla u|^p dx \right)^{1/p} |x-y|^{\alpha}.$$

$\Rightarrow$ local (weighted) Hölder regularity for homogeneous p-Laplace

J. Lewis. Regularity of the derivatives of solutions to certain degenerate elliptic equations. Indiana Univ. Math. J., 32(6):849–858, 1983.

Local Hölder regularity of the inhomog. p-Laplace

We can transfer the local Hölder regularity from the homogeneous case to the inhomogeneous p-Laplace equation:

Theorem (Kuusi,Mingione 2013; Diening,Kaplicky,Schwarzacher)

Ω,d,p as before. Let

$$\alpha^* = \sup\{\alpha : \textit{Theorem of Lewis holds including the estimates}\}.$$

Then for u with $\Delta_p u = f \in C^{1,\beta(\alpha)}$:

$$u \textit{ is locally } \alpha\textit{-Hölder continuous for } \alpha < \min(\alpha^*, 1/(p-1)).$$

Analog estimates hold for local Hölder-seminorm of u.

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Ω,d,p as before. Let

$$\alpha^* = \sup\{\alpha : \textit{Theorem of Lewis holds including the estimates}\}.$$

Then for u with $\Delta_p u = f \in C^{1,\beta(\alpha)}$:

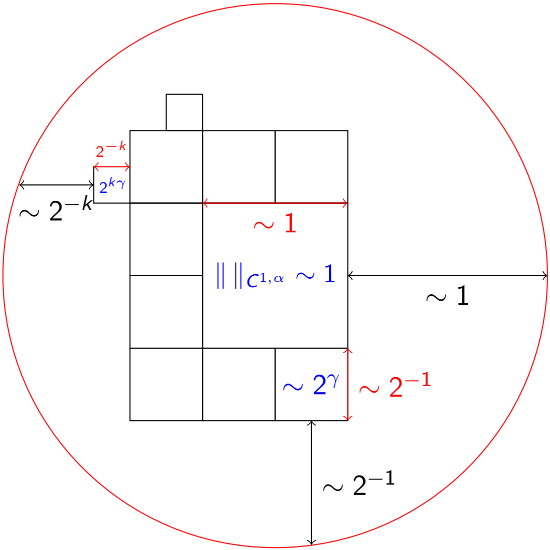
u is locally α-Hölder continuous for $\alpha < \min(\alpha^*, 1/(p-1))$.

Analog estimates hold for local Hölder-seminorm of u.

Problem: $\alpha^* \in (0,1]$ is unknown for $d \geq 3$. (later: case $d = 2$)

T. Kuusi and G. Mingione. Guide to Nonlinear Potential Estimates. Bull. Math. Sci, 4(1):1–82, 2014.

Locally weighted Hölder spaces $C^{\ell,\alpha}_{\gamma,\mathrm{loc}}(\Omega)$



$C^{\ell,\alpha}_{\gamma,\mathrm{loc}}(\Omega)\ldots$ Hölder space, locally weighted, with

$\ell\ldots$ number of derivatives

$\alpha\ldots$ Hölder exponent of derivatives of order ℓ

$\gamma\ldots$ growth of Hölder exp. with distance to $\partial\Omega$

Locally weighted Hölder spaces $C^{\ell,\alpha}_{\gamma,\mathrm{loc}}(\Omega)$ (ii)

Definition (Locally weighted Hölder spaces)

K compact subset of Ω, δ_K distance to $\partial\Omega$, $\mathcal{K}$ family of compact subsets of Ω, $g \in C^\ell(\Omega)$, set

$$|g|_{C^{\ell,\alpha}(K)} := \sum_{|\nu|=\ell} \sup_{\substack{x,y\in K,\\ x\neq y}} \frac{|\partial^\nu g(x) - \partial^\nu g(y)|}{|x-y|^\alpha},$$

$$|g|_{C^{1,\alpha}_{\gamma,\mathrm{loc}}(\mathcal{K})} := \sup_{K\in\mathcal{K}} \delta_K^\gamma \, |g|_{C^{\ell,\alpha}(K)} < \infty,$$

$$C^{\ell,\alpha}_{\gamma,\mathrm{loc}}(\Omega;\mathcal{K}) = \left\{ g \in C^\ell(\Omega) : |g|_{C^{\ell,\alpha}_{\gamma,\mathrm{loc}}(\mathcal{K})} < \infty \right\}.$$

- $\mathcal{K}$ shall be the set of all $B(x_0, r)$ such that $B(x_0, 64r) \subset \Omega$.
- This definition ($\ell = 1$) is perfectly adapted to Lewis' Theorem.

Local Hölder regularity of the p-Laplace

Although the optimal local Hölder regularity of the solution of the p-Poisson is unknown ($d \geq 3$), we can estimate γ by Lewis' Theorem

$$\max_{x,y \in B(x_0,r)} |\nabla u(x) - \nabla u(y)| \leq C \cdot r^{-\alpha} \left(\fint_{B(x_0,32r)} |\nabla u|^p dx \right)^{1/p} |x-y|^\alpha$$

$$\leq C \cdot r^{-\alpha} \left(\fint_{B(x_0,32r)} |\nabla u|^q dx \right)^{1/q} |x-y|^\alpha, \quad p \leq q.$$

$$\leq C \cdot r^{-\alpha-d/q} \cdot \|\nabla u \,|\, L_q(\Omega)\| \cdot |x-y|^\alpha.$$

Hence, using the result of Ebmeyer

$$|\nabla u| \in L_q(\Omega) \text{ for } q < \frac{pd}{d-1},$$

we are allowed to choose

$$\gamma = \alpha + (d-1)/p + \varepsilon \text{ for all } \varepsilon > 0.$$

The case $d = 2$: Hölder regularity of the p-Poisson (i)

Theorem (Lindgren, Lindqvist 2013; (DDHSW 2014))

$\Omega \subset \mathbb{R}^2$ bounded polygonal domain, $1 < p < \infty$, $f \in L_\infty(\Omega)$. If $\Delta_p u = f$, $u = 0$ on $\partial\Omega$, then u is locally α-Hölder continuous for all

$$\alpha < \begin{cases} 1, & \text{if } \ 1 < p \leq 2, \\ \frac{1}{p-1}, & \text{if } \ 2 < p < \infty. \end{cases}$$

Furthermore, for the same α's, it holds

$$u \in C^{1,\alpha}_{\gamma,\mathrm{loc}}(\Omega) \text{ for } \gamma = \alpha + 1/p + \varepsilon.$$

- The regularity $\frac{1}{p-1}$ is a natural bound, take $v(x) = |x|^{p/(p-1)}$.
- homogen. case: Iwaniec, Manfredi (1989) proved $u \in C^{\ell,\alpha}_{\mathrm{loc}}(\Omega)$ with

$$\ell + \alpha = 1 + \frac{1}{6}\left(1 + \frac{1}{p-1} + \sqrt{1 + \frac{14}{p-1} + \frac{1}{(p-1)^2}}\right) > \max\left(2, \frac{p}{p-1}\right)$$

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From $B^s_{p,p}(\Omega)$ and $C^{\ell,\alpha}_{\gamma,\mathrm{loc}}(\Omega)$ to $B^\sigma_{\tau,\tau}(\Omega)$

Theorem (Dahlke, Diening, Hartmann, S., Weimar(DDHSW) '14)

$\Omega \subset \mathbb{R}^d$ *bound. Lipschitz dom.,* $d \geq 2$, $s > 0$, $1 < p < \infty$, $\alpha \in (0,1]$,

$$\sigma^* = \begin{cases} \ell + \alpha, & \text{if} \quad 0 < \gamma < \frac{\ell+\alpha}{d} + \frac{1}{p}, \\ \frac{d}{d-1}\left(\ell + \alpha + \frac{1}{p} - \gamma\right), & \text{if} \quad \frac{\ell+\alpha}{d} + \frac{1}{p} \leq \gamma < \ell + \alpha + \frac{1}{p}, \end{cases}$$

then for all

$$0 < \sigma < \min\left\{\sigma^*, \frac{d}{d-1} s\right\} \qquad \text{and} \qquad \frac{1}{\tau} = \frac{\sigma}{d} + \frac{1}{p}$$

we have the continuous embedding

$$B^s_{p,p}(\Omega) \cap C^{\ell,\alpha}_{\gamma,\mathrm{loc}}(\Omega) \hookrightarrow B^\sigma_{\tau,\tau}(\Omega).$$

If γ not too bad and local Hölder regularity $\ell + \alpha$ is higher than Sobolev regularity s, Besov regularity σ is higher than Sobolev reg. !

The case $d = 2$: Besov regularity of the p-Poisson

1. By Ebmeyer's result

$$u \in \begin{cases} B^{3/2-\varepsilon}_{p,p}(\Omega), & \text{if } 1 < p \leq 2, \\ B^{1+1/p-\varepsilon}_{p,p}(\Omega), & \text{if } p \geq 2, \end{cases}$$

2. Lindgren, Lindqvist:

$$u \in C^{1,\alpha}_{\gamma,\mathrm{loc}}(\Omega),\ \gamma = \alpha + 1/p + \varepsilon,\ \alpha < \begin{cases} 1, & \text{if } 1 < p \leq 2, \\ \frac{1}{p-1}, & \text{if } 2 < p < \infty. \end{cases}$$

3. γ not too bad? $\alpha + \frac{1}{p} + \varepsilon = \gamma \overset{?}{<} \frac{\ell+\alpha}{d} + \frac{1}{p} = \frac{1+\alpha}{2} + \frac{1}{p}$? Yes, $\alpha < 1$
4. General embedding theorem, $\frac{1}{\tau} = \frac{\sigma}{d} + \frac{1}{p}$,

$$u \in B^{\sigma}_{\tau,\tau}(\Omega) \text{ for all } \sigma < \begin{cases} 2, & \text{if } 1 < p \leq 2, \\ 1 + \frac{1}{p-1}, & \text{if } 2 < p < \infty. \end{cases}$$

Summary: Besov regularity of the p-Poisson

- For $d = 2$ results on Besov regularity beat Sobolev regularity:

$$\text{it holds} \begin{cases} 2 > 3/2, & \text{if} \quad 1 < p \leq 2, \\ 1 + \frac{1}{p-1} > 1 + \frac{1}{p} & \text{if} \quad 2 < p < \infty. \end{cases}$$

- For $d \geq 3$ the optimal α is unknown, known: $\alpha \to 0$ for $p \to \infty$
- For $d \geq 3$ to beat Sobolev regularity we need

$$\alpha > \begin{cases} \frac{1}{2}, & \text{if } 1 < p < 2, \\ \frac{1}{p}, & \text{if } p > 2, \end{cases}$$

and γ not too large depending on d. This implies

$$p \in (p_d, \infty) \text{ with } p_d \to \infty \text{ for } d \to \infty.$$

E. Lindgren and P. Lindqvist. Regularity of the p-poisson equation in the plane. arXiv:1311.6795v2, 2013.

T. Iwaniec and J. Manfredi. Regularity of p-harmonic functions on the plane. Rev. Mat. Iberoamericana, 5(1-2):119, 1989.

Open problems

- $d = 2$, can one do better, in dependency of the angles of the boundary?
- Is the chosen γ optimal?
- non-polyhedral domains
- measure Besov regularity in L_q for the p-Laplace ($q \neq p$)
- bring the $F^{s,\mathrm{rloc}}_{p,q}(\Omega)$ spaces into play...

work in progress. . .

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Thank you for your attention

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