How to steer high-dimensional Cucker-Smale systems to consensus using low-dimensional information only

Benjamin Scharf

Technische Universität München,
Department of Mathematics,
Applied Numerical Analysis

benjamin.scharf@ma.tum.de

TUM-IAS Workshop on Novel Numerical Methods, July 30, 2013

Overview

The Cucker-Smale model - Introdruction

... a dynamical system used to describe the nature of a group of moving agents, i. e. birds, formation/evolution of languages etc.

$$\begin{cases} \dot{x}_i(t) = v_i(t) \\ \dot{v}_i(t) = \frac{1}{N}\sum_{j=1}^{N} a(\|x_j(t) - x_i(t)\|) \cdot (v_j(t) - v_i(t)), \end{cases}$$

where $x_1, \ldots, x_N, v_1, \ldots, v_N \in \mathbb{R}^d$ with given initial values at time 0 and a is a *non-increasing pos. Lipschitz function*. Cucker and Smale:

$$a(x) = \frac{K}{(\sigma^2 + x^2)^\beta}, \; K, \sigma > 0, \beta \geq 0$$

Ref.: *F. Cucker and S. Smale. Emergent behavior in flocks. IEEE Trans. Automat. Control, 52(5):852–862, 2007.*

F. Cucker and S. Smale. On the mathematics of emergence. Jpn. J. Math., 2(1):197–227, 2007.

The Cucker-Smale model - Main parameters

To measure the distances of the particles as well as their velocities we introduce:

$$X(t) := \frac{1}{2N^2}\sum_{i,j=1}^{N}\|x_i(t)-x_j(t)\|^2 = \frac{1}{N}\sum_{i=1}^{N}\|x_i(t)-\overline{x}(t)\|^2 = \overline{x^2}-\overline{x}^2,$$

$$V(t) := \frac{1}{2N^2}\sum_{i,j=1}^{N}\|v_i(t)-v_j(t)\|^2 = \frac{1}{N}\sum_{i=1}^{N}\|v_i(t)-\overline{v}(t)\|^2 = \overline{v^2}-\overline{v}^2$$

$v_i^{\perp}(t) := v_i(t) - \overline{v}(t).$

The Cucker-Smale model - Main parameters

To measure the distances of the particles as well as their velocities we introduce:

$$X(t) := \frac{1}{2N^2} \sum_{i,j=1}^{N} \|x_i(t) - x_j(t)\|^2 = \frac{1}{N} \sum_{i=1}^{N} \|x_i(t) - \overline{x}(t)\|^2 = \overline{x^2} - \overline{x}^2,$$

$$V(t) := \frac{1}{2N^2} \sum_{i,j=1}^{N} \|v_i(t) - v_j(t)\|^2 = \frac{1}{N} \sum_{i=1}^{N} \|v_i(t) - \overline{v}(t)\|^2 = \overline{v^2} - \overline{v}^2$$

$v_i^{\perp}(t) := v_i(t) - \overline{v}(t).$

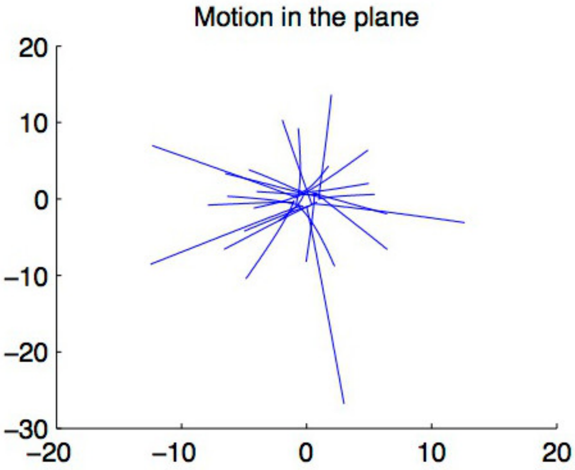
The main question is: Does the system tend to consensus?

? $\lim_{t\to\infty} v_i(t) = \overline{v}$ or equivalently $\lim_{t\to\infty} v_i^{\perp}(t) = 0$ resp. $\lim_{t\to\infty} V(t) = 0$?

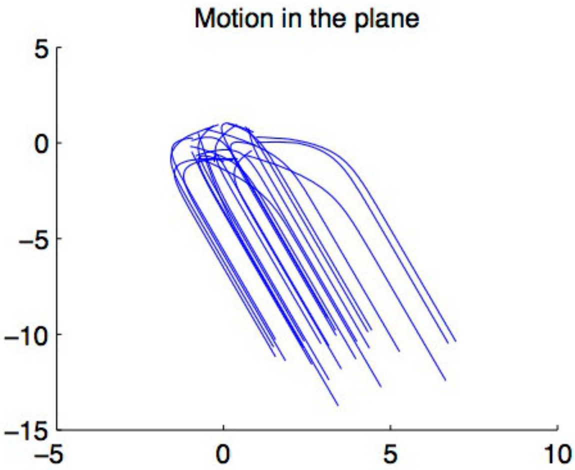
This would imply: The system moves as a swarm, i. e.

$$x(t) \approx x(t_0) + (t - t_0)\overline{v}$$

The Cucker-Smale model - Explosion



The Cucker-Smale model - Consensus



The Cucker-Smale model - Consensus

Lemma (Lyapunov functional behaviour)

$$\frac{d}{dt}V(t) \leq a\left(\sqrt{2NX(t)}\right)\sqrt{V(t)} \text{ as long as V(t)} > 0$$

Hence: If $X(t)$ is bounded, the system tends to consensus.

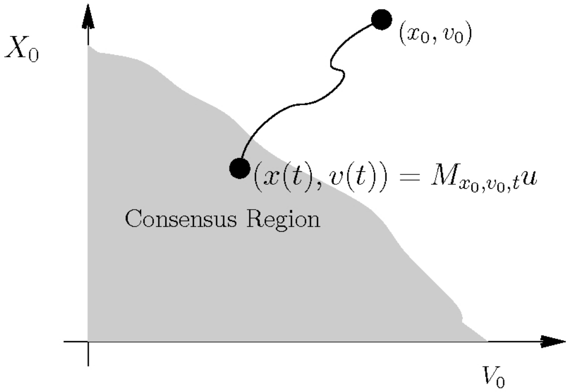
Theorem (Ha, Ha, Kim 2010)

$$\text{If } \int_{\sqrt{X(0)}}^{\infty} a\left(\sqrt{2Nr}\right)\, dr \geq \sqrt{V(0)}, \text{ then } \lim_{t\to\infty} V(t) = 0.$$

Table of contents

The consensus manifold

Idea: If we are not in the consensus manifold, use (sparse) control



Ref.: *M. Caponigro, M. Fornasier, B. Piccoli and E. Trelat. Sparse stabilization and control of the Cucker-Smale model. 2012.*

Sparsely Controlled Cucker-Smale system

Goal: Stear the system to the consensus area using control and then stop the control.

Minimize the time to consensus and the number of agents to act on:

$$\begin{cases} \dot{x}_i(t) = v_i(t) \\ \dot{v}_i(t) = \frac{1}{N}\sum_{j=1}^{N} a(\|x_j(t) - x_i(t)\|) \cdot (v_j(t) - v_i(t)) + u_i(t). \end{cases}$$

with $\ell_1^N - \ell_2^d$-norm constraint (compare to compressed sensing)

$$\sum_{i=1}^{N} \|u_i(t)\|_2 \leq \Theta.$$

Observe: The control only acts on the most stubborn guy! (shepherd dog/Schäfarhund strategy)

How to construct controls - Sample and Hold

Sample and Hold idea: First construct solutions with controls constant on intervals $[k\tau, (k+1)\tau]$ - time-sparse controls.

Observe: The optimality criterion (decay of $V(t)$) doesn't hold anymore.

How to construct controls - Sample and Hold

Sample and Hold idea: First construct solutions with controls constant on intervals $[k\tau, (k+1)\tau]$ - time-sparse controls.

Observe: The optimality criterion (decay of $V(t)$) doesn't hold anymore.

Theorem (Caponigro, Fornasier, Piccoli, Trelat)

For every Θ (constraint size) there exists $\tau_0 > 0$ such that for all sampling times $\tau \in [0, \tau_0]$ the sampling solution of the controlled Cucker-Smale system reaches the consensus region in finite time.

How to construct controls - picture

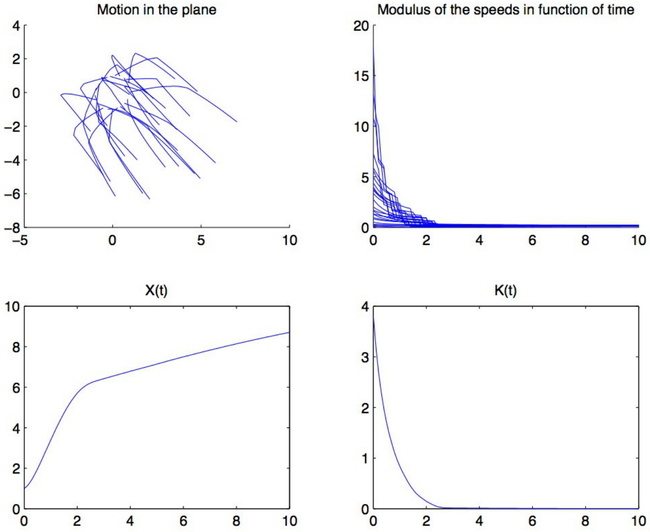


Table of contents

Introduction

The case $N \to \infty$ (large number of agents) is widely considered in the literature:

- locations, velocities $\Rightarrow$ density distributions
- dynamical system, ODE $\Rightarrow$ PDE

Ref. (e. g.): *M. Fornasier and F. Solombrino. Mean-field optimal control. 2013.*

Introduction

The case $N \to \infty$ (large number of agents) is widely considered in the literature:

- locations, velocities $\Rightarrow$ density distributions
- dynamical system, ODE $\Rightarrow$ PDE

Ref. (e. g.): *M. Fornasier and F. Solombrino. Mean-field optimal control. 2013.*

The case $d \to \infty$ (high dimension/many coordinates/variables) is of our interest.

Example: Social movement (panic), Financial movement

x not locations, more variables/state of the system
(Health, pulse, strength; situation on the market, IFO-Index etc.)
v describes the movement towards consensus

$\Rightarrow$ Goal: Panic prevention, Black Swan prevention

Reduction of the dimension of Cucker-Smale

Goal: Reduction of the Cucker-Smale-like models without control

$$
\begin{aligned}
M\dot{v}_i(t) &= \frac{1}{N}\sum_{j=1}^{N} a(\|x_j(t) - x_i(t)\|) \cdot (Mv_j(t) - Mv_i(t)) \\
&\sim \frac{1}{N}\sum_{j=1}^{N} a(\|Mx_j(t) - Mx_i(t)\|) \cdot (Mv_j(t) - Mv_i(t)) \,.
\end{aligned}
$$

Idea: Consider low-dimensional Cucker-Smale system in $\mathbb{R}^k$ with $(y_0, w_0) = (Mx_0, Mv_0)$ as initial values.

Ref.: *M. Fornasier, J. Haskovec and J. Vybiral. Particle systems and kinetic equations modeling interacting agents in high dimension, 2011.*

Johnson-Lindenstrauss matrices

Lemma (Johnson-Lindenstrauss matrices (JLM))

Let $x_1, \ldots, x_N$ be points in $\mathbb{R}^d$. Given $\varepsilon > 0$, there exists a constant

$$k_0 = \mathcal{O}(\varepsilon^{-2} \log N),$$

such that for all integers $k \geq k_0$ there exists a $k \times d$ matrix M for which

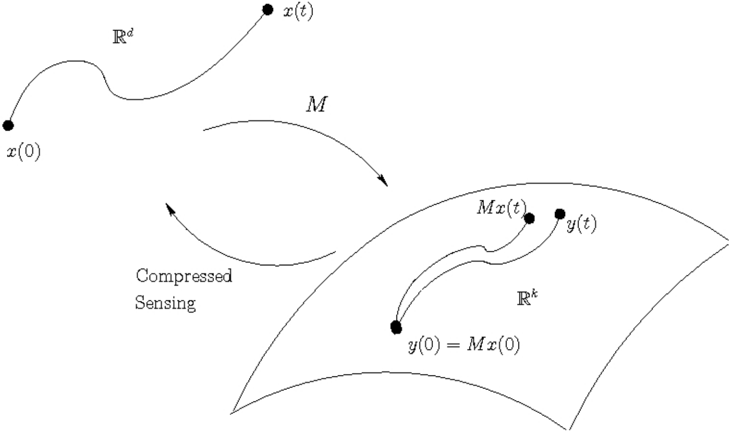
$$(1-\varepsilon)\|x_i\|^2 \leq \|Mx_i\|^2 \leq (1+\varepsilon)\|x_i\|^2, \text{ for all } i = 1, \ldots, N.$$

Fornasier, Haskovec and Vybiral show:

JLM-projection of the high-dimensional system stays close to the low-dimensional system

or: first project, then dynamics $\sim$ first dynamics, then project

Low-dim. and projection of high-dim. stay close



The plan – dimension reduction with control

$$\begin{cases} \dot{x}_i(t) = v_i(t) \\ \dot{v}_i(t) = \frac{1}{N}\sum_{j=1}^{N} a(\|x_j(t) - x_i(t)\|)\cdot(v_j(t) - v_i(t)) + u_i(t). \end{cases}$$

1. Project the high-dimensional initial values with JLM M to low-dimension
2. Choose the index of sparse control $(u_i \neq 0)$ from the low-dimensional system and apply it to the high-dimensional system
3. Show: If the systems stay close to each other, then either
   - both systems are in consensus or
   - the control is reasonable for both systems (decay of $V(t)$ is fast enough)

The main result

Theorem (Bongini, Fornasier, Scharf)

Let $M \in \mathbb{R}^{k \times d}$ be a continuous Johnson-Lindenstrauss matrix for the distances of $x(t), v(t)$ with ε and δ sufficiently (very, very) small. Choose the sparse control index according to the low k-dimensional Cucker-Smale system with initial values $(Mx(0), Mv(0))$.
Then for every Θ (constraint) and $\tau < \tau_0$ the sampling solution of the so controlled high-dimensional Cucker-Smale system reaches the consensus region in finite time.

Remarks:

- ε and δ (and everything else) do not depend on d, but heavily on N
- If $\Theta >> 0$ and is constant with N, then ε can be choosen s. t.

$$\varepsilon \sim c \frac{1}{N e^{N/c}}$$

There is another problem...

In the theorem we suppose M is a JLM for the distances of $x(t), v(t)$, but:
The initial values of the low-dimensional system and hence the controls depend on M $\Rightarrow$ Right now, as a worst case scenario one has to take all possible trajectories into account.

There is another problem...

In the theorem we suppose M is a JLM for the distances of $x(t), v(t)$, but:
The initial values of the low-dimensional system and hence the controls depend on M $\Rightarrow$ Right now, as a worst case scenario one has to take all possible trajectories into account.

Room for improvement!

Thank you for your attention