How to steer high-dimensional Cucker-Smale systems to consensus using low-dimensional information only

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TUM-IAS Workshop on Novel Numerical Methods, July 30, 2013
Overview

Introduction and classical results of the Cucker-Smale model
  The general Cucker-Smale model
  Pictures
  Results on consensus

Steering Cucker-Smale model to consensus using sparse control

Dimension reduction of the Cucker-Smale model
  Introduction high dimension
  Dimension reduction and Johnson-Lindenstrauss matrices (JLM)
  Can we use low-dimension information to find the right control?
The Cucker-Smale model - Introduction

... a dynamical system used to describe the nature of a group of moving agents, i.e. birds, formation/evolution of languages etc.

\[
\begin{cases}
\dot{x}_i(t) = v_i(t) \\
\dot{v}_i(t) = \frac{1}{N} \sum_{j=1}^{N} a(\|x_j(t) - x_i(t)\|) \cdot (v_j(t) - v_i(t)),
\end{cases}
\]

where \( x_1, \ldots, x_N, v_1, \ldots, v_N \in \mathbb{R}^d \) with given initial values at time 0 and \( a \) is a non-increasing pos. Lipschitz function. Cucker and Smale:

\[
a(x) = \frac{K}{(\sigma^2 + x^2)^\beta}, \quad K, \sigma > 0, \beta \geq 0
\]


The Cucker-Smale model - Main parameters

To measure the distances of the particles as well as their velocities we introduce:

\[ X(t) := \frac{1}{2N^2} \sum_{i,j=1}^{N} \|x_i(t) - x_j(t)\|^2 = \frac{1}{N} \sum_{i=1}^{N} \|x_i(t) - \bar{x}(t)\|^2 = \bar{x}^2 - \bar{x}^2, \]

\[ V(t) := \frac{1}{2N^2} \sum_{i,j=1}^{N} \|v_i(t) - v_j(t)\|^2 = \frac{1}{N} \sum_{i=1}^{N} \|v_i(t) - \bar{v}(t)\|^2 = \bar{v}^2 - \bar{v}^2 \]

\[ v_i^\perp(t) := v_i(t) - \bar{v}(t). \]
The Cucker-Smale model - Main parameters

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\[ V(t) := \frac{1}{2N^2} \sum_{i,j=1}^{N} \| v_i(t) - v_j(t) \|^2 = \frac{1}{N} \sum_{i=1}^{N} \| v_i(t) - \bar{v}(t) \|^2 = \bar{v}^2 - \bar{v}^2 \]

\[ v_i^\perp(t) := v_i(t) - \bar{v}(t). \]

The main question is: Does the system tend to consensus?

? \( \lim_{t \to \infty} v_i(t) = \bar{v} \) or equivalently \( \lim_{t \to \infty} v_i^\perp(t) = 0 \) resp. \( \lim_{t \to \infty} V(t) = 0 \)?

This would imply: The system moves as a swarm, i.e.

\[ x(t) \approx x(t_0) + (t - t_0)\bar{v} \]
The Cucker-Smale model - Explosion

Motion in the plane
The Cucker-Smale model - Consensus

Motion in the plane
The Cucker-Smale model - Consensus

**Lemma (Lyapunov functional behaviour)**

\[ \frac{d}{dt} V(t) \leq a \left( \sqrt{2N}X(t) \right) \sqrt{V(t)} \text{ as long as } V(t) > 0 \]

Hence: If \( X(t) \) is bounded, the system tends to consensus.

**Theorem (Ha, Ha, Kim 2010)**

If \( \int_{X(0)}^{\infty} a \left( \sqrt{2Nr} \right) \, dr \geq \sqrt{V(0)} \), then \( \lim_{t \to \infty} V(t) = 0 \).
Table of contents

Introduction and classical results of the Cucker-Smale model
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  Pictures
  Results on consensus

Steering Cucker-Smale model to consensus using sparse control

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The consensus manifold

Idea: If we are not in the consensus manifold, use (sparse) control

Sparsely Controlled Cucker-Smale system

Goal: Stear the system to the consensus area using control and then stop the control.

Minimize the time to consensus and the number of agents to act on:

\[
\begin{align*}
\dot{x}_i(t) &= v_i(t) \\
\dot{v}_i(t) &= \frac{1}{N} \sum_{j=1}^{N} a(\|x_j(t) - x_i(t)\|) \cdot (v_j(t) - v_i(t)) + u_i(t).
\end{align*}
\]

with $\ell_1^N - \ell_2^d$-norm constraint (compare to compressed sensing)

\[
\sum_{i=1}^{N} \|u_i(t)\|_2 \leq \Theta.
\]

Observe: The control only acts on the most stubborn guy! (shepherd dog/Schäfarhund strategy)
How to construct controls - Sample and Hold

Sample and Hold idea: First construct solutions with controls constant on intervals $[k\tau, (k + 1)\tau]$ - time-sparse controls.

Observe: The optimality criterion (decay of $V(t)$) doesn’t hold anymore.
How to construct controls - Sample and Hold

Sample and Hold idea: First construct solutions with controls constant on intervals \([k\tau, (k+1)\tau]\) - time-sparse controls.

Observe: The optimality criterion (decay of \(V(t)\)) doesn’t hold anymore.

**Theorem (Caponigro, Fornasier, Piccoli, Trelat)**

For every \(\Theta\) (constraint size) there exists \(\tau_0 > 0\) such that for all sampling times \(\tau \in [0, \tau_0]\) the sampling solution of the controlled Cucker-Smale system reaches the consensus region in finite time.
How to construct controls - picture

Motion in the plane

Modulus of the speeds in function of time

$X(t)$

$K(t)$
Table of contents

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Introduction

The case \( N \to \infty \) (large number of agents) is widely considered in the literature:

- locations, velocities \( \Rightarrow \) density distributions
- dynamical system, ODE \( \Rightarrow \) PDE

Ref. (e.g.): M. Fornasier and F. Solombrino. Mean-field optimal control. 2013.
Introduction

The case $N \to \infty$ (large number of agents) is widely considered in the literature:

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The case $d \to \infty$ (high dimension/many coordinates/variables) is of our interest.

**Example:** Social movement (panic), Financial movement

$x$ not locations, more variables/state of the system
(Health, pulse, strength; situation on the market, IFO-Index etc.)

\(v\) describes the movement towards consensus

$\Rightarrow$ Goal: Panic prevention, Black Swan prevention
Reduction of the dimension of Cucker-Smale

Goal: Reduction of the Cucker-Smale-like models without control

\[ \dot{Mv}_i(t) = \frac{1}{N} \sum_{j=1}^{N} a(\|x_j(t) - x_i(t)\|) \cdot (Mv_j(t) - Mv_i(t)) \]

\[ \sim \frac{1}{N} \sum_{j=1}^{N} a(\|Mx_j(t) - Mx_i(t)\|) \cdot (Mv_j(t) - Mv_i(t)). \]

Idea: Consider low-dimensional Cucker-Smale system in \( \mathbb{R}^k \) with \((y_0, w_0) = (Mx_0, Mv_0)\) as initial values.

Ref.: M. Fornasier, J. Haskovec and J. Vybiral. Particle systems and kinetic equations modeling interacting agents in high dimension, 2011.
Johnson-Lindenstrauss matrices

Lemma (Johnson-Lindenstrauss matrices (JLM))
Let \( x_1, \ldots, x_N \) be points in \( \mathbb{R}^d \). Given \( \varepsilon > 0 \), there exists a constant
\[
k_0 = O(\varepsilon^{-2} \log N),
\]
such that for all integers \( k \geq k_0 \) there exists a \( k \times d \) matrix \( M \) for which
\[
(1 - \varepsilon)\|x_i\|^2 \leq \|Mx_i\|^2 \leq (1 + \varepsilon)\|x_i\|^2, \text{ for all } i = 1, \ldots, N.
\]

Fornasier, Haskovec and Vybiral show:

JLM-projection of the high-dimensional system stays close to the low-dimensional system

or: first project, then dynamics \( \sim \) first dynamics, then project
Low-dim. and projection of high-dim. stay close

\[ x(0) \rightarrow \mathbb{R}^d \rightarrow M \rightarrow \mathbb{R}^k \]

Compressed Sensing

\[ y(0) = Mx(0) \]
The plan – dimension reduction with control

\[
\begin{align*}
\dot{x}_i(t) &= v_i(t) \\
\dot{v}_i(t) &= \frac{1}{N} \sum_{j=1}^{N} a(\|x_j(t) - x_i(t)\|) \cdot (v_j(t) - v_i(t)) + u_i(t).
\end{align*}
\]

1. Project the high-dimensional initial values with JLM $M$ to low-dimension

2. Choose the index of sparse control ($u_i \neq 0$) from the low-dimensional system and apply it to the high-dimensional system

3. Show: If the systems stay close to each other, then either
   - both systems are in consensus or
   - the control is reasonable for both systems (decay of $V(t)$ is fast enough)
The main result

Theorem (Bongini, Fornasier, Scharf)

Let $M \in \mathbb{R}^{k \times d}$ be a continuous Johnson-Lindenstrauss matrix for the distances of $x(t), v(t)$ with $\varepsilon$ and $\delta$ sufficiently (very, very) small. Choose the sparse control index according to the low $k$-dimensional Cucker-Smale system with initial values $(Mx(0), Mv(0))$. Then for every $\Theta$ (constraint) and $\tau < \tau_0$ the sampling solution of the so controlled high-dimensional Cucker-Smale system reaches the consensus region in finite time.

Remarks:
- $\varepsilon$ and $\delta$ (and everything else) do not depend on $d$, but heavily on $N$
- If $\Theta \gg 0$ and is constant with $N$, then $\varepsilon$ can be choosen s. t.
  $$\varepsilon \sim c \frac{1}{Ne^{N/c}}$$
There is another problem...

In the theorem we suppose $M$ is a JLM for the distances of $x(t), v(t)$, but:

The initial values of the low-dimensional system and hence the controls depend on $M$ ⇒ Right now, as a worst case scenario one has to take all possible trajectories into account.
There is another problem...

In the theorem we suppose $M$ is a JLM for the distances of $x(t), v(t)$, but:

The initial values of the low-dimensional system and hence the controls depend on $M \Rightarrow$ Right now, as a worst case scenario one has to take all possible trajectories into account.

Room for improvement!

Thank you for your attention