How to steer high-dimensional Cucker-Smale systems to consensus using low-dimensional information only

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The Cucker-Smale model - Introduction

. . . a dynamical system used to describe the nature of a group of moving agents, i. e. birds, but also the formation/evolution of languages etc.

$$\begin{cases} \dot{x}_i(t) = v_i(t) \\ \dot{v}_i(t) = \frac{1}{N} \sum_{j=1}^{N} a(\|x_j(t) - x_i(t)\|) \cdot (v_j(t) - v_i(t)), \end{cases}$$

where $x_1, \dots, x_N, v_1, \dots, v_N \in \mathbb{R}^d$ with given initial values at 0 and a is a *non-increasing positive Lipschitz function*. Example of Cucker and Smale:

$$a(x) = \frac{K}{(\sigma^2 + x^2)^\beta}, \, K, \sigma > 0, \beta \geq 0$$

References:

F. Cucker and S. Smale. Emergent behavior in flocks. IEEE Trans. Automat. Control, 52(5):852–862, 2007.

F. Cucker and S. Smale. On the mathematics of emergence. Jpn. J. Math., 2(1):197–227, 2007.

The Cucker-Smale model - First observations

First observations:

1. Bigger difference between velocities $\Rightarrow$ bigger change of velocity
2. Bigger distance of particles $\Rightarrow$ smaller influence on the change of velocity
3. Mean velocity $\overline{v}(t) = \frac{1}{N}\sum_{j=1}^{N} v_i(t)$ is a constant of the system
4. Rotation of the start parameters $x_1(0), \ldots, x_N(0), v_1(0), \ldots, v_N(0)$ results in rotation of the system

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4. Rotation of the start parameters $x_1(0), \dots, x_N(0), v_1(0), \dots, v_N(0)$ results in rotation of the system
5. We can rewrite the system as

$$\begin{cases} \dot{x} = v \\ \dot{v} = -L_x v, \end{cases} \qquad \text{with } L_x = (a_{ij})_{i,j=1}^{N} \text{ and } a_{ii} = \sum_{j \neq i} -a_{ij},$$

symmetric L_x, $a_{ii} \geq 0$, $a_{ij} \leq 0$ and hence L_x positive semi-definite.

The Cucker-Smale model - Main parameters

To measure the distances of the particles as well as their velocities we introduce:

$$X(t) := \frac{1}{2N^2} \sum_{i,j=1}^{N} \|x_i(t) - x_j(t)\|^2 = \frac{1}{N} \sum_{i=1}^{N} \|x_i(t) - \overline{x}(t)\|^2 = \overline{x^2} - \overline{x}^2,$$

$$V(t) := \frac{1}{2N^2} \sum_{i,j=1}^{N} \|v_i(t) - v_j(t)\|^2 = \frac{1}{N} \sum_{i=1}^{N} \|v_i(t) - \overline{v}(t)\|^2 = \overline{v^2} - \overline{v}^2$$

$$v_i^{\perp}(t) := v_i(t) - \overline{v}(t).$$

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$$v_i^{\perp}(t) := v_i(t)-\overline{v}(t).$$

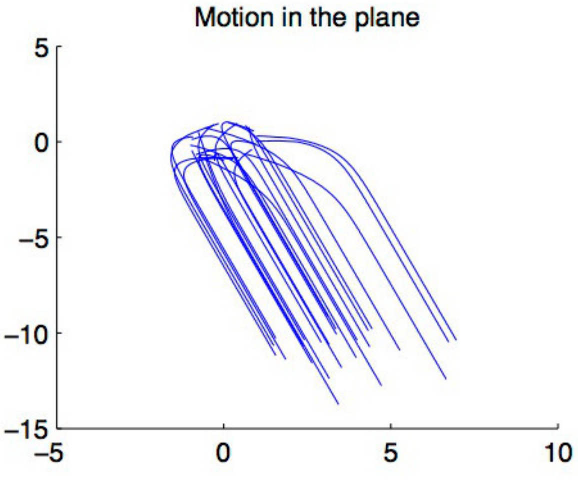
The main question is: Does the systems tend to consensus?

$$?\lim_{t\to\infty} v_i(t) = \overline{v} \text{ or equivalently } \lim_{t\to\infty} v_i^{\perp}(t) = 0 \text{ resp. } \lim_{t\to\infty} V(t) = 0?$$

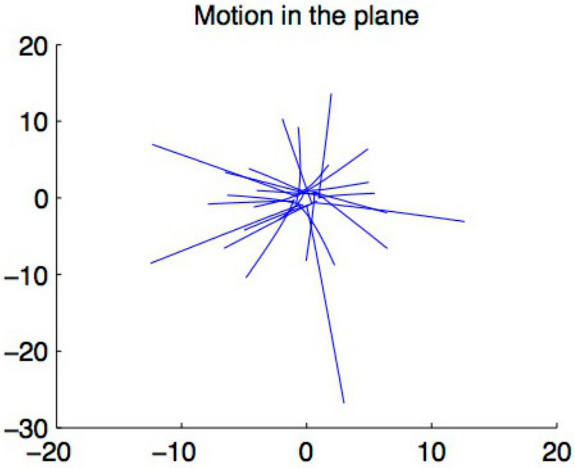
This would imply: The system moves as a swarm, i. e.

$$x(t) \approx x(t_0) + (t-t_0)\overline{v}$$

The Cucker-Smale model - Consensus



The Cucker-Smale model - Explosion



The Cucker-Smale model - Consensus

First observe

$$\frac{d}{dt}V(t) = \frac{1}{N}\sum_{i=1}^{N}\frac{d}{dt}\|v_i(t)\|^2 = \frac{2}{N}\sum_{i=1}^{N} < v_i(t), \dot{v}_i(t) > = - < v, L_x v >_{\mathbb{R}^{d\times n}}$$

$$= -\frac{1}{N^2}\sum_{i,j=1}^{N} a(\|x_j(t) - x_i(t)\|) \cdot \|v_j(t) - v_i(t)\|^2.$$

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$$= -\frac{1}{N^2}\sum_{i,j=1}^{N} a(\|x_j(t) - x_i(t)\|) \cdot \|v_j(t) - v_i(t)\|^2.$$

Lemma (Lyapunov functional behaviour)

$$\frac{d}{dt}V(t) \le a\left(\sqrt{2NX(t)}\right)\sqrt{V(t)} \text{ as long as } V(t) > 0$$

Hence: If $X(t)$ is bounded, the system tends to consensus.

The Cucker-Smale model - Consensus (ii)

Theorem (Ha, Ha, Kim 2010)

$$\textit{If } \int_{\sqrt{X(0)}}^{\infty} a\left(\sqrt{2N}r\right)\, dr \geq \sqrt{V(0)}, \textit{ then } \lim_{t\to\infty} V(t) = 0.$$

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Remarks:

1. a not integrable $\Rightarrow$ the system tends to consensus independent of the start parameters
2. Otherwise: If the distance of the actors is not too large resp. the starting velocities are not too different, the system tends to consensus

Example

Classical C.-S. distance $a(x) = \frac{1}{(1+x^2)^\beta}$

- $\beta \leq 1/2$: always consensus (strong enough forces)
- $\beta > 1/2$: depends on the initial values

The Cucker-Smale model - Consensus (iii)

Example

Two agents, $\beta = 1$, consider there distance $x = x_1 - x_2$ and difference of velocity $v = v_1 - v_2$:

$$\dot{x} = v, \quad \dot{v} = -\frac{v}{1+x^2}$$

with initial distance $x(0) = x_0$ and diff. of velocities $v(0) = v_0 > 0$. Hence $0 < v(t) \leq v_0$ since $|v(t)|$ is decreasing and $v(t') = 0 \Rightarrow v(t) = 0, t \geq t'$.

$$\text{This yields} \quad v(t) - v_0 = -\arctan x(t) + \arctan x_0.$$

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This yields $\quad v(t) - v_0 = -\arctan x(t) + \arctan x_0.$

- $\arctan x_0 + v_0 < \pi/2 \Rightarrow x(t)$ bounded:
  $a)$ $\arctan x(t) \leq \arctan x(t) + v(t) < \pi/2$
  $b)$ $\arctan x(t) \geq (v_0 - v(t)) + \arctan x_0 \geq \arctan x_0$
- $\arctan x_0 + v_0 = v(t) + \arctan x(t) = \pi/2 \Rightarrow v(t) \downarrow 0$ or $x(t)$ bound.
- $\arctan x_0 + v_0 = \pi/2 + \varepsilon \Rightarrow v(t) + \arctan x(t) = \pi/2 + \varepsilon \Rightarrow v(t) \geq \varepsilon$

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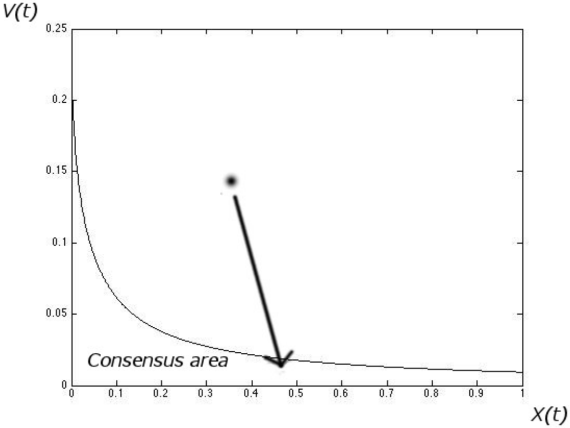
The consensus manifold

Idea: If we are not in the consensus manifold, infer (sparse) control



Ref.: *M. Caponigro, M. Fornasier, B. Piccoli and E. Trelat. Sparse stabilization and control of the Cucker-Smale model. submitted, 2012.*

Sparsely Controlled Cucker-Smale system

Goal: Stear the system to the consensus area using control and then stop the control. Minimize the necessary "control steps" - minimize the time to consensus and the number of agents to act on:

$$\begin{cases} \dot{x}_i(t) = v_i(t) \\ \dot{v}_i(t) = \frac{1}{N}\sum_{j=1}^{N} a(\|x_j(t) - x_i(t)\|) \cdot (v_j(t) - v_i(t)) + u_i(t). \end{cases}$$

with $\ell_1^N - \ell_2^d$-norm constraint (compare to compressed sensing)

$$\sum_{i=1}^{N} \|u_i(t)\|_2 \leq \Theta.$$

Observe: $\overline{v}$ is not constant anymore.

Maximizing the decay of $V(t)$

Maximizing the decay of $V(t)$ with respect to the $\ell_1^N - \ell_2^d$-norm constraint leads to the so-called shepherd dog (Schäferhund) strategy:

$$\frac{d}{dt}V(t) = \frac{d}{dt} < v - \overline{v}, v - \overline{v} >= 2 < \frac{d}{dt}v^\perp, v^\perp >$$
$$= 2 < \dot{v}, v^\perp >= - < L_x v, v > + < u, v^\perp >$$
$$\Rightarrow u_i = \begin{cases} -M\frac{v_i^\perp}{\|v_i^\perp\|} & \text{if } i \text{ is first } i : \|v_i^\perp\| = \max_{j=1,\dots,N} \|v_j^\perp\| \\ 0 & \text{otherwise} \end{cases}$$

is (one/the) maximizer under $\sum_{i=1}^N \|u_i(t)\|_2 \leq \Theta$.

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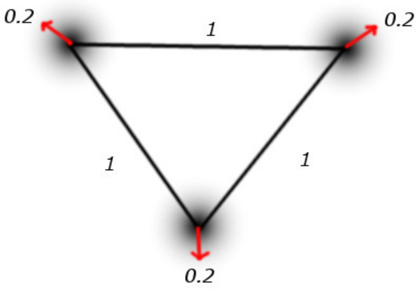
$$\begin{aligned} \frac{d}{dt}V(t) &= \frac{d}{dt} < v - \overline{v}, v - \overline{v} >= 2 < \frac{d}{dt}v^{\perp}, v^{\perp} > \\ &= 2 < \dot{v}, v^{\perp} >= - < L_x v, v > + < u, v^{\perp} > \\ \Rightarrow u_i &= \begin{cases} -M \frac{v_i^{\perp}}{\|v_i^{\perp}\|} & \text{if } i \text{ is first } i : \|v_i^{\perp}\| = \max_{j=1,\dots,N} \|v_j^{\perp}\| \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

is (one/the) maximizer under $\sum_{i=1}^N \|u_i(t)\|_2 \leq \Theta$.

The control only acts on the most stubborn guy!

How to construct controls - Paradox of switching controls

The controls are defined pointwisely and influence the future. The following example shows the problem:



Assume u_1 is active for $[0, t] \Rightarrow v_1$ is nearer to $\overline{v}$ at $t/2$ than v_2 ⨳

How to construct controls - Sample and Hold

Sample and Hold idea: First construct solutions with controls constant on intervals $[k\tau, (k+1)\tau]$ - time-sparse controls.
Recursive construction of the sampling solution:
As long as we are not in the consensus manifold at $t = k\tau$ solve

$$\dot{z}(t) = f(z(t), u(z(k\tau))), \quad t \in [k\tau, (k+1)\tau]$$

with initial value $z(k\tau)$ and $u(k\tau)$ chosen as before.
Observe: The optimality criterion (decay of $V(t)$) doesn't hold anymore.

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Observe: The optimality criterion (decay of $V(t)$) doesn't hold anymore.

Theorem (Caponigro, Fornasier, Piccoli, Trelat)

For every Θ (constraint size) there exists $\tau_0 > 0$ such that for all sampling times $\tau \in [0, \tau_0]$ the sampling solution of the controlled Cucker-Smale system reaches the consensus region in finite time.

How to construct controls - Filippov solution

Convergence: Take the solutions x_τ with respect to the sampling time τ and let $\tau \to 0$. Prove that z_τ converges to a z in a suitable way.

$$z_\tau = z_0 + \int_0^t f(z_\tau(s)) + u_\tau(z_\tau(s))\, ds.$$

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1. z_τ are bounded on finite intervals (Gronwall estimate)
2. z_τ are equicontinuous (equi-Lipschitz)
3. z_τ converges by Arzela-Ascoli in $\mathcal{C}$ to $z \in Lip$.
4. $$\int_0^t u_\tau(z_\tau(s))\, ds \to y(t)$$
5. Since u_τ are bounded, y is absolutely continuous, can be written as
$$y(t) = \int_0^t u(s)\, ds.$$
6. Density argument: $u_\tau(z(\tau)) \to u$ weakly in L_1

How to construct controls - Filippov solution

A deeper argument shows: The limit control u is of the form

$$u_i = \begin{cases} -\alpha_i \frac{v_i^\perp}{\|v_i^\perp\|} & \text{, if } \|v_i^\perp\| = \max_{j=1,\dots,N} \|v_j^\perp\| \\ 0 & \text{, otherwise} \end{cases}$$

until reaching consensus region and minimizes the decay of $V(t)$: It is possibly not sparse (Example!).

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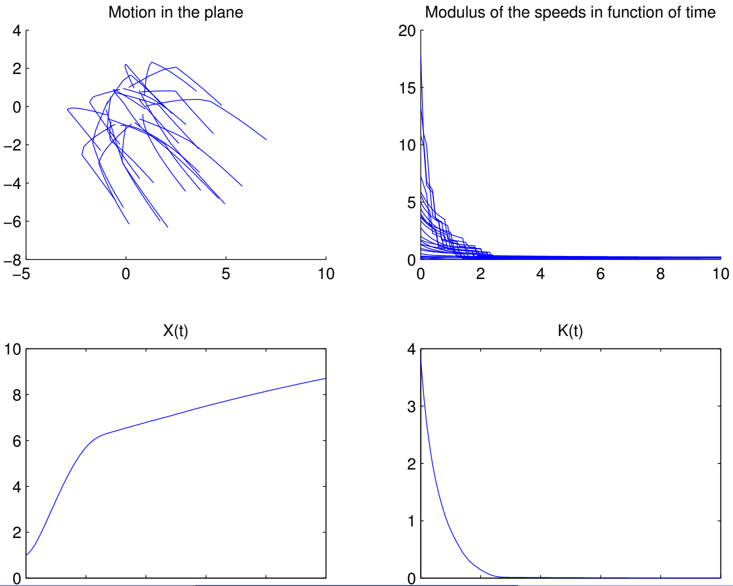
Remarks and open problems:

- The time to consensus can be estimated from above depending on $X(0), V(0)$ and the constraint Θ
- Greedy minimization may not be optimal
- What is the minimal time to consensus?
- How much control interactions are necessary?

How to construct controls - picture



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Introduction

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- locations, velocities $\Rightarrow$ density distributions
- dynamical system, ODE $\Rightarrow$ PDE

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The case $d \to \infty$ (high dimension/many coordinates/variables) is of our interest.
Example: Social movement (panic), Financial movement
x not locations, more variables/state of the system
(Health, pulse, strength; situation on the market, IFO-Index etc.)

v describes the movement towards consensus

$\Rightarrow$ Goal: Panic prevention, Black Swan prevention

Johnson-Lindenstrauss matrices

The main tool is the dimension reduction by Johnson-Lindenstrauss:

Lemma (Johnson-Lindenstrauss matrices (JLM))

Let $x_1, \dots, x_N$ be points in $\mathbb{R}^d$. Given $\varepsilon > 0$, there exists a constant

$$k_0 = \mathcal{O}(\varepsilon^{-2} \log N),$$

such that for all integers $k \geq k_0$ there exists a $k \times d$ matrix M for which

$$(1-\varepsilon)\|x_i\|^2 \leq \|Mx_i\|^2 \leq (1+\varepsilon)\|x_i\|^2, \text{ for all } i = 1, \dots, N.$$

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Remarks:

- M can be understood as a low-dimensional replacement for a projection onto $span\{x_1, \ldots, x_N\}$
- k_0 does not depend on the dimension, only logarithmically on the number of points N, usely $N \sim d^\alpha \Rightarrow$ logarithmically on d
- the construction of JLM uses random matrices, no deterministic construction known

How to reduce the dimension of the Cucker-Smale model

M. Fornasier, J. Haskovec and J. Vybiral. Particle systems and kinetic equations modeling interacting agents in high dimension, 2011.

$\Rightarrow$ Reduction of the Cucker-Smale-like models without control.

$$\dot{Mv_i}(t) = \frac{1}{N}\sum_{j=1}^{N} a(\|x_j(t) - x_i(t)\|) \cdot (Mv_j(t) - Mv_i(t))$$

$$\sim \frac{1}{N}\sum_{j=1}^{N} a(\|Mx_j(t) - Mx_i(t)\|) \cdot (Mv_j(t) - Mv_i(t)) \, .$$

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Idea: Consider a low-dimensional Cucker-Smale system in $\mathbb{R}^k$ (low dimension JLM) with $(y_0, w_0) = (Mx_0, Mv_0)$ as initial values. They show:

JLM-projection of the high-dimensional system stays close to the low-dimensional system

or: first project, then dynamics $\sim$ first dynamics, then project

First tool: A continuous Johnson-Lindenstrauss lemma

Lemma (Bongini, Fornasier, Scharf (BFS))

Let $\varphi : [0,1] \to \mathbb{R}^d$ *be a Lipschitz function (bound* L_φ*),* $0 < \varepsilon < \varepsilon' < 1$*,* $\delta > 0$ *and* M *be a Johnson-Lindenstrauss matrix in* $\mathbb{R}^{k \times d}$ *for*

$$N \geq L_\varphi \frac{6d/k}{\delta(\varepsilon' - \varepsilon)}$$

points with high probability. Then for every $t \in [0,1]$ *one of the following holds (with the same high probability):*

$$(1-\varepsilon')\|\varphi(t)\| \leq \|M\varphi(t)\| \leq (1+\varepsilon')\|\varphi(t)\|$$

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$$\|\varphi(t)\| \leq \delta \text{ and } \|M\varphi(t)\| \leq \delta.$$

Remarks: The original lemma of Fornasier, Haskovec, Vybiral assumed that φ has bounded curvature which is not given in the Cucker-Smale case.

What is the plan?

$$\begin{cases} \dot{x}_i(t) = v_i(t) \\ \dot{v}_i(t) = \frac{1}{N}\sum_{j=1}^{N} a(\|x_j(t) - x_i(t)\|) \cdot (v_j(t) - v_i(t)) + u_i(t). \end{cases}$$

1. Project the high-dimensional initial values with JLM M to low-dimension
2. Choose the index of sparse control ($u_i \neq 0$) from the low-dimensional system and apply it to the high-dimensional system
3. Show: If the systems stay close to each other, then
   - either both systems are in consensus or
   - the control is reasonable for both systems (decay of $V(t)$ is fast enough)

Norm estimates for high-dimensional control

Lemma (High-dimensional control is legit, BFS)

Le M be a Johnson-Lindenstrauss matrix with $\varepsilon = 1/2$ and δ for the points a_i. Assume $\|Ma_i - b_i\| \leq \delta$. Let i be the smallest index such that $\|b_i\| \geq \|b_j\|$ and

$$A := \frac{1}{N}\sum_{j=1}^{N}\|a_j\|^2 \text{ and } B := \frac{1}{N}\sum_{j=1}^{N}\|b_j\|^2.$$

If $\sqrt{B} \geq 2\delta$, then (c indep. of d, N)

$$\|a_i\| \geq \frac{\|b_i\|}{4}, \quad \|a_i\| \geq c \cdot \sqrt{A} \text{ and } B \leq 4NA.$$

If $\sqrt{B} \leq 2\delta$, then (C indep. of d, N)

$$\sqrt{A} \leq C\delta.$$

The main result

Theorem (BFS)

Let $M \in \mathbb{R}^{k \times d}$ be a continuous Johnson-Lindenstrauss matrix for the distances of $x(t), v(t)$ with ε and δ sufficiently (very, very) small. Choose the sparse control index according to the low k-dimensional Cucker-Smale system with initial values $(Mx(0), Mv(0))$.
Then for every Θ (constraint) and $\tau < \tau_0$ the sampling solution of the so controlled high-dimensional Cucker-Smale system reaches the consensus region in finite time.

Remarks:

- ε and δ (and everything else) do not depend on d, but heavily on N
- If $\Theta >> 0$ and is constant with N, then ε can be choosen such that:

$$\varepsilon \sim c \frac{1}{Ne^{N/c}}$$

There is another problem...

In the theorem we suppose M is a JLM for the distances of $x(t), v(t)$, but: The initial values of the low-dimensional system and hence the controls depend on M

$\Rightarrow$ the high-dimensional system depends on M: Vicious circle

Solutions so far:

- take M as JLM for all possible trajectories, in principal $N^{T/\tau}$ possibilities where T is the time until consensus $\Rightarrow$ we have to estimate the exponent T/τ, problematic
- use different matrices of the same dimension for every choice of the control (at $k\tau$) $\Rightarrow$ a lot of matrix-vector multiplications in dimension d, since T/τ depends on N at least linearly right now

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Thank you for your attention!