Wavelets for reinforced function spaces on cellular domains

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Domains in $\mathbb{R}^n$ (i)

(i) Let $2 \leq n$. A special Lipschitz (C^∞-) domain in $\mathbb{R}^n$ is the collection of all points $x = (x', x_n)$ with $x' \in \mathbb{R}^{n-1}$ and $h(x') < x_n$, where h is a Lipschitz (bounded C^∞-) function.

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(ii) Let $2 \leq n$. A bounded Lipschitz (C^∞-) domain is a bounded domain Ω where the boundary $\Gamma = \partial\Omega$ can be covered by finitely many open balls B_j centred at Γ such that $B_j \cap \Omega = B_j \cap \Omega_j$, where Ω_j are rotations of suitable special Lipschitz (C^∞-) domains.

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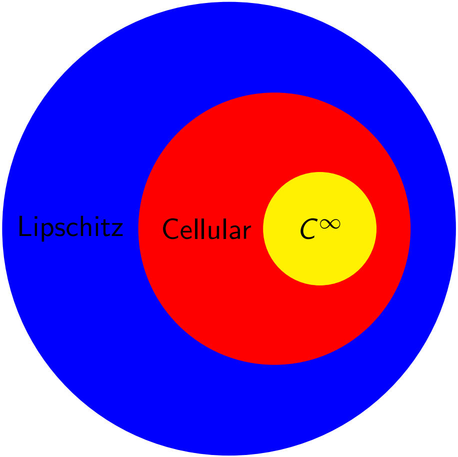
(iii) Let $2 \leq n$. A domain Ω in $\mathbb{R}$ is called cellular if it is a bounded Lipschitz domain which can be represented as

$$\Omega = \left(\bigcup_{\ell=1}^{L} \bar{\Omega}_\ell \right)^\circ \text{ with } \Omega_\ell \cap \Omega_{\ell'} = \emptyset \text{ if } \ell \neq \ell'$$

such that each Ω_ℓ is diffeomorphic to a polyhedron (cube).

Domains in $\mathbb{R}^n$ (ii)



The definition of $F^s_{p,q}(\mathbb{R})$

Let $\{\varphi_j\}_{j=0}^{\infty}$ be a resolution of unity. Let $0 < p < \infty$, $0 < q \leq \infty$ and $s \in \mathbb{R}$. For $f \in \mathcal{S}'(\mathbb{R}^n)$ we define

$$\|f|F^s_{p,q}(\mathbb{R}^n))\|^{\varphi} := \left\| \left(\sum_{j=0}^{\infty} 2^{jsq} |(\varphi_j \hat{f})^{\vee}|^q \right)^{\frac{1}{q}} \middle| L_p(\mathbb{R}^n) \right\|$$

(modified in case $q = \infty$) and

$$F^s_{p,q}(\mathbb{R}^n) := \left\{ f \in \mathcal{S}'(\mathbb{R}^n) : \|f|F^s_{p,q}(\mathbb{R}^n)\|^{\varphi} < \infty \right\}.$$

Then $(F^s_{p,q}(\mathbb{R}^n), \|\cdot|F^s_{p,q}(\mathbb{R}^n)\|^{\varphi})$ is a quasi-Banach space. It does not depend on the choice of the resolution of unity $\{\varphi_j\}_{j=0}^{\infty}$ in the sense of equivalent norms. So we denote it shortly by $F^s_{p,q}(\mathbb{R}^n)$.

The definition of $F^s_{p,q}(\Omega)$, $\tilde{F}^s_{p,q}(\Omega)$ and $\mathring{F}^s_{p,q}(\Omega)$ (i)

Let Ω be an open set in $\mathbb{R}^n$. Denote by $g|\Omega \in D'(\Omega)$ its restriction to Ω, hence $(g|\Omega)(\varphi) = g(\varphi)$ for $\varphi \in D(\Omega)$.

$$F^s_{p,q}(\Omega) := \{f \in D'(\Omega) : f = g|\Omega \text{ for some } g \in F^s_{p,q}(\mathbb{R}^n)\},$$
$$\|f|F^s_{p,q}(\Omega)\| = \inf \|g|F^s_{p,q}(\mathbb{R}^n)\|,$$

where the infimum is taken over all $g \in F^s_{p,q}(\mathbb{R}^n)$ with $g|\Omega = f$.

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where the infimum is taken over all $g \in F^s_{p,q}(\mathbb{R}^n)$ with $g|\Omega = f$. Moreover, let

$$\tilde{F}^s_{p,q}(\bar{\Omega}) := \{f \in F^s_{p,q}(\mathbb{R}^n) : \mathit{supp}\, f \subset \bar{\Omega}\}$$

with the quasi-norm from $F^s_{p,q}(\mathbb{R}^n)$. Then

$$\tilde{F}^s_{p,q}(\Omega) := \{f \in D'(\Omega) : f = g|\Omega \text{ for some } g \in \tilde{F}^s_{p,q}(\bar{\Omega})\},$$
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where the infimum is taken over all $g \in \tilde{F}^s_{p,q}(\bar{\Omega})$ with $g|\Omega = f$.

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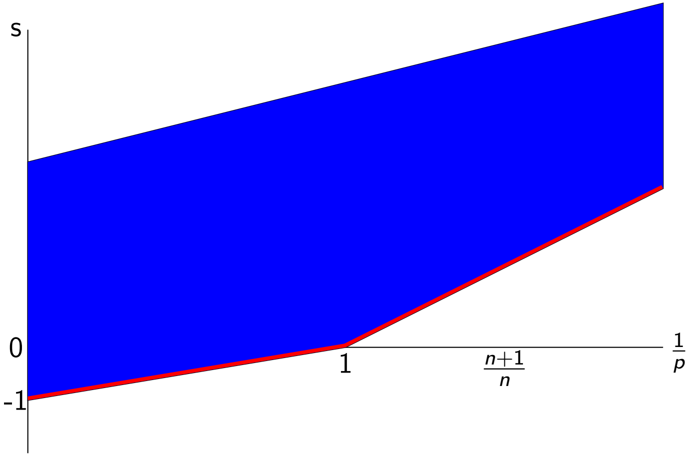

The definition of $F^s_{p,q}(\Omega)$, $\tilde{F}^s_{p,q}(\Omega)$ and $\mathring{F}^s_{p,q}(\Omega)$ (iii)

Furthermore, let $\mathring{F}^s_{p,q}(\Omega)$ be the completion of $D(\Omega)$ with respect to $\| \cdot |F^s_{p,q}(\Omega)\|$.

Theorem (The starting stripe - T08, Prop. 6.13.)

Let Ω be a cellular domain and let

$$0 < p < \infty, 0 < q < \infty, \max\left(n\left(\frac{1}{p} - 1\right), \frac{1}{p} - 1\right) < s < \frac{1}{p}$$

Then

$$F^s_{p,q}(\Omega) = \mathring{F}^s_{p,q}(\Omega) = \tilde{F}^s_{p,q}(\Omega).$$

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Then

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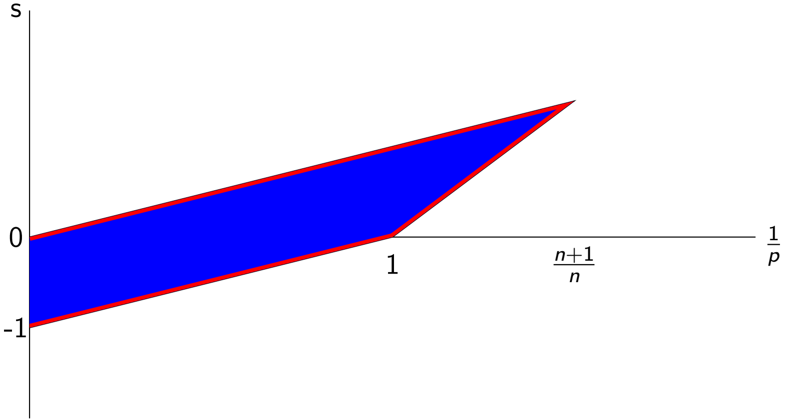
Proof: Use that χ_Ω is a pointwise multiplier in these spaces.

The starting stripe



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Wavelets for function spaces on domains Ω (i)

From now on let Ω be a cellular domain and $\Gamma = \partial\Omega$. Let

$$\mathbb{Z}^{\Omega} := \left\{ x_\ell^j \in \Omega : j \in \mathbb{N}_0, \ell = 1, \ldots, N_j \right\},$$

(typically $N_j \sim 2^{jn}$), such that for some $c_1 > 0$

$$|x_\ell^j - x_{\ell'}^j| \geq c_1 2^{-j},\ j \in \mathbb{N}_0,\ \ell \neq \ell'.$$

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$$|x_\ell^j - x_{\ell'}^j| \geq c_1 2^{-j},\ j \in \mathbb{N}_0,\ \ell \neq \ell'.$$

We can introduce the usual sequence spaces $f_{p,q}^s(\mathbb{Z}^\Omega)$ adapted to $\mathbb{Z}^\Omega$: Let

$$\|\lambda | f_{p,q}^s(\mathbb{Z}^\Omega)\| := \left\| \left(\sum_{j=0}^{\infty} \sum_{\ell=1}^{N_j} 2^{jsq} |\lambda_\ell^j \cdot \chi_{j,\ell}(\cdot)|^q \right)^{\frac{1}{q}} | L_p(\Omega) \right\|,$$

where $\chi_{j,\ell}$ is the characteristic function of $B(x_\ell^j, c2^{-j})$ or $\bar{\Omega} \cap B(x_\ell^j, c2^{-j})$ with an arbitrary constant $c > 0$.

Wavelets for function spaces on domains Ω (ii)

Let $u \in \mathbb{N}_0$.
Then

$$\Phi = \left\{ \Phi_\ell^j : j \in \mathbb{N}_0, \ell = 1, \ldots, N_j \right\} \subset C^u(\Omega)$$

is called a u-wavelet system in $\bar{\Omega}$ (adapted to $\mathbb{Z}^\Omega$) if it fulfils

- support conditions: For some $c_3 > 0$ it holds

$$supp\, \Phi_\ell^j \subset B(x_\ell^j, c_3 2^{-j}) \cap \bar{\Omega},\ j \in \mathbb{N}_0,\ \ell = 1, \ldots, N_j,$$

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- derivative conditions: For some $c_4 > 0$ and all $\alpha \in \mathbb{N}_0^n$ with $0 \leq |\alpha| \leq u$ let

$$\left| D^\alpha \Phi_\ell^j(x) \right| \leq c_4 2^{j\frac{n}{2} + j|\alpha|}, x \in \Omega,\ j \in \mathbb{N}_0,\ \ell = 1, \ldots, N_j.$$

Wavelets for function spaces on domains Ω (iii)

Additionally, the u-wavelet system is called oscillating if it fulfils

- (substitute) moment conditions: Let c_5 and $c_6 < c_7$ be constants such that

$$\operatorname{dist}(B(x_\ell^0, c_3), \Gamma) \geq c_6, \text{ for } \ell = 1, \ldots, N_0 \text{ and}$$

$$\left| \int_\Omega \psi(x) \Phi_\ell^j(x)\, dx \right| \leq c_5 2^{-j\frac{n}{2} - ju} \, \|\psi | C^u(\Omega)\| \text{ for all } \psi \in C^u(\Omega)$$

for all j and ℓ with $dist(B(x_\ell^j, c_3), \Gamma) \notin \left(c_6 2^{-j}, c_7 2^{-j}\right)$.

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  for all j and ℓ with $dist(B(x_\ell^j, c_3), \Gamma) \notin \left(c_6 2^{-j}, c_7 2^{-j}\right)$.

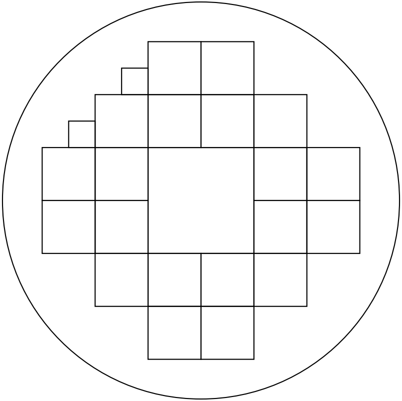
An oscillating u-wavelet system is called interior if it fulfils

- (further) interior support conditions, namely

$$\text{dist}(B(x_\ell^j, c_3 2^{-j}), \Gamma) \geq c_6 2^{-j}, \, j \in \mathbb{N}_0, \, \ell = 1, \ldots, N_j.$$

Interior wavelets on Ω

The support of the first interior wavelets (on a ball):



Interior wavelet bases in $L_2(\Omega)$ - the starting point

Theorem (T08 - Theorem 2.33)

Let Ω be an arbitrary domain in $\mathbb{R}^n$. For any $u \in \mathbb{N}_0$ there is a

$$\Phi = \left\{ \Phi^j_\ell : j \in \mathbb{N}_0, \ell = 1, \ldots, N_j \right\} \subset C^u(\Omega)$$

which is

1. *an orthonormal basis in $L_2(\Omega)$,*
2. *an interior u-wavelet system*

simultaneously.

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For $u = 0$ one can take the Haar Wavelet suitably restricted to Ω.

Riesz bases on arbitrary domains $\Omega \subset \mathbb{R}^n$

Definition (Riesz basis)

An [oscillating]{interior} u-wavelet system $\Phi = \{\Phi_r^j\}$ is called [oscillating] {interior} u-Riesz basis for X with a suitable sequence space x :⇔

1. An element $f \in D'(\Omega)$ belongs to X if, and only if, it can be represented as
$$f = \sum_{j=0}^{\infty} \sum_{r=1}^{N_j} \lambda_r^j(f) 2^{-\frac{jn}{2}} \Phi_r^j, \lambda \in x \tag{1}$$
with unconditional convergence in X.
2. The representation (1) is unique and the coefficient mappings
$$\lambda_r^j : f \mapsto \lambda_r^j(f)$$
are linear and continuous functionals on X.
3. Furthermore $f \mapsto \{\lambda_r^j(f)\}$ is an isomorphic map of X onto x.

Interior wavelet bases in $\tilde{F}^s_{p,q}(\Omega)$ and $L_p(\Omega)$

Theorem (T08 - Theorem 2.36, Prop. 3.10, Prop. 3.13)

Let Ω be a Lipschitz (E-thick) domain in $\mathbb{R}^n$. Then the interior u-wavelet basis Φ in $L_2(\Omega)$ is an interior Riesz basis for

1. $\tilde{F}^0_{p,2}(\Omega) = L_p(\Omega)$, $1 < p < \infty$, $u > 0$ *(or Haar wavelet)*
2. $\tilde{F}^s_{p,q}(\Omega)$, $0 < p < \infty$, $0 < q < \infty$, $u > s > \sigma_{p,q}$.
3. $F^s_{p,q}(\Omega)$, $0 < p < \infty$, $0 < q < \infty$, $u > \sigma_{p,q} - s$, $s < 0$

with the sequence space $f^s_{p,q}(\mathbb{Z}^\Omega)$ and

$$\lambda^j_r(f) = 2^{jn/2}(f, \Phi^j_r).$$

Interior wavelet bases in the starting stripe

Corollary (T08 - Theorem 5.43)

Let Ω be a cellular domain and let $1 \le p < \infty$, $0 < q < \infty$. Then for large enough $u \in \mathbb{N}_0$

$$\Phi = \left\{ \Phi^j_\ell : j \in \mathbb{N}_0, \ell = 1, \ldots, N_j \right\} \subset C^u(\Omega)$$

are interior Riesz bases for $F^s_{p,q}(\Omega)$ in the starting stripe - more general, even for

$$-\infty < s < \frac{1}{p}.$$

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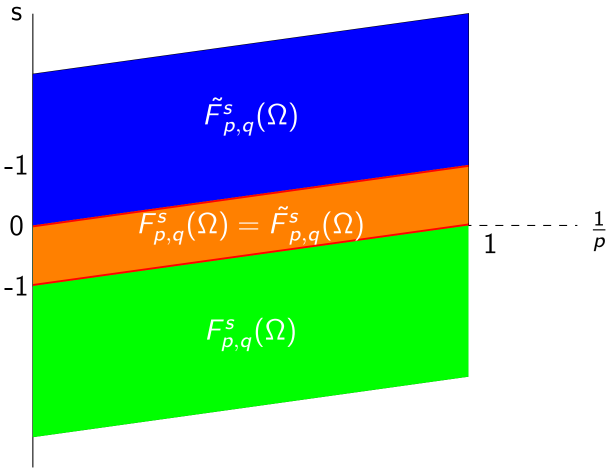
Proof idea: By $F^s_{p,q}(\Omega) = \tilde{F}^s_{p,q}(\Omega)$ the cases with $s > \sigma_q$ follow from the theorems before. The rest is a matter of complex interpolation (s and q).

Note: For $s > \frac{1}{p}$ there are no **interior** u-wavelet systems for $F^s_{p,q}(\Omega)$ because of boundary values. We have to find "non-interior" wavelet syst.

Interior wavelet bases - summary for $1 \leq p < \infty$



Traces on the boundary of cubes (i)

Let $Q = \{x \in \mathbb{R}^n : x = (x_1, \ldots, x_n), 0 < x_m < 1, m = 1, \ldots, n)$. The boundary $\Gamma = \partial Q$ of Q can be represented as

$$\Gamma = \bigcup_{\ell=0}^{n-1} \Gamma_\ell \text{ with } \Gamma_\ell \cap \Gamma_{\ell'} = \emptyset \text{ for } \ell \neq \ell',$$

where $\Gamma_\ell = \bigcup_{j=0}^{n_\ell} Q_{\ell,j}$ consists of all ℓ-dimensional faces $Q_{\ell,j}$ of Q, which are disjoint cubes of dimension ℓ.

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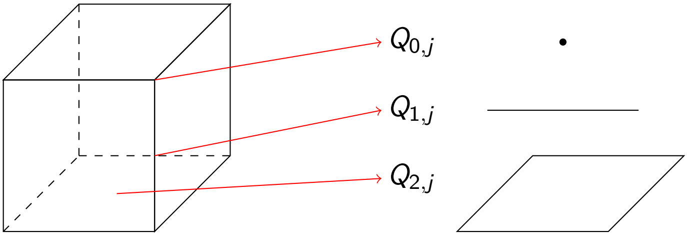

Traces on the boundary of cubes (ii)

Let $tr_{\ell,j}$ be the restriction of $f \in A^s_{p,q}(\mathbb{R}^n)$ to $Q_{\ell,j}$ and

$$tr^r_\ell : f \mapsto TR^r_\ell(f) := \prod \left\{ tr_{\ell,j}\, D^\alpha_\gamma f : |\alpha| \le r,\, j = 0, \ldots, n_\ell \right\},$$

where only derivatives perpendicular to $Q_{\ell,j}$ are admitted.

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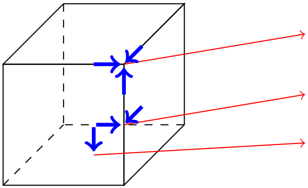
where only derivatives perpendicular to $Q_{\ell,j}$ are admitted. Then we consider the composite mapping for $\bar{r} = \left(r^{\ell_0}, \ldots r^{n-1} \right)$ with $\ell_0 \le n-1$ and $r^\ell = \lceil s - \frac{n-\ell}{p} \rceil$-1:

$$tr^{\bar{r}}_\Gamma : f \mapsto \prod_{\ell=\ell_0}^{n-1} TR^{\bar{r}}_\ell(f).$$



$Q_{0,j}$: 3 directions, $\lceil s - \frac{3}{p} \rceil$ derivatives

$Q_{1,j}$: 2 directions, $\lceil s - \frac{2}{p} \rceil$ derivatives

$Q_{2,j}$: 1 direction, $\lceil s - \frac{1}{p} \rceil$ derivatives

Traces on the boundary of cubes (iii)

Proposition

Let

$$1 \le p < \infty, 0 < q < \infty, s > \frac{1}{p}.$$

Let $\ell_0 = 0$ *if* $s > \frac{n}{p}$*. Otherwise* $\ell_0 \in \mathbb{N}$ *is chosen such that*

$$0 < s - \frac{n - \ell_0}{p} \le \frac{1}{p}.$$

Then

$$tr_{\Gamma}^{\bar{r}} : F^s_{p,q}(Q) \mapsto \prod_{\ell=\ell_0}^{n-1} \prod_{j=0}^{N_\ell} \prod_{|\alpha| \le r^\ell} F_{p,p}^{s - \frac{n-\ell}{p} - |\alpha|}(Q_{\ell,j}).$$

Traces on the boundary of cubes (iv)

Proposition

Let

$$1 \le p < \infty, 0 < q < \infty, s > \frac{1}{p} \text{ and } s - \frac{k}{p} \notin \mathbb{N}_0 \text{ for } k = 1, \dots, n.$$

Let $\ell_0 = 0$ *if* $s > \frac{n}{p}$. *Otherwise* $\ell_0 \in \mathbb{N}$ *is chosen such that*

$$0 < s - \frac{n - \ell_0}{p} < \frac{1}{p}.$$

Then

$$\tilde{F}^s_{p,q}(Q) = \left\{ f \in F^s_{p,q}(Q) : tr^{\bar{r}}_{\Gamma} = 0 \right\}.$$

This is a generalization of the $(n-1)$-dimensional result

$$\tilde{F}^s_{p,q}(\Omega) = \mathring{F}^s_{p,q}(\Omega) = \{ f \in F^s_{p,q}(\Omega) : \mathrm{tr}_\Gamma = 0 \}$$

for $s > \frac{1}{p}$ and $s - \frac{1}{p} \notin \mathbb{N}$, where tr_Γ is the composite of the (existing) traces on $\Gamma = \partial\Omega$ of a C^∞-domain Ω.

Extension operators for the boundary of cubes (i)

Proposition

Let p, q, s, ℓ as in the theorems before and additionally $u > s$. Then there is a **wavelet-friendly** *extension operator*

$$Ext_\ell^{r,u} : \prod_{\substack{\alpha \ perp. \ Q_{\ell,j} \\ \|\alpha| \leq r}} \tilde{F}_{p,p}^{s - \frac{n-\ell}{p} - |\alpha|}(Q_{\ell,j}) \mapsto F_{p,q}^s(Q).$$

It holds

$$tr_\ell^r \circ Ext_\ell^{r,u} = id \text{ on the left space.}$$

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This theorem also works for $s - \frac{k}{p} \in \mathbb{N}$.

Extension operators for the boundary of cubes (ii)

Proposition

Furthermore,

$$F^s_{p,q}(Q) = \tilde{F}^s_{p,q}(Q) \times Ext^{\bar{r},u}_{\Gamma} \prod_{\ell=\ell_0}^{n-1} \prod_{j=0}^{N_\ell} \prod_{|\alpha| \le r^\ell} \tilde{F}^{s-\frac{n-\ell}{p}-|\alpha|}_{p,p}(Q_{\ell,j}).$$

Extension operators for the boundary of cubes (ii)

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Proof: Use

$$\tilde{F}^s_{p,q}(Q) = \left\{ f \in F^s_{p,q}(Q) : tr^{\bar{r}}_{\Gamma} = 0 \right\}.$$

and

$$tr^r_\ell \left(f - Ext^{r,u}_\ell \circ tr^r_\ell \right) = 0.$$

Wavelets for cubes

Theorem (T08 - Theorem 6.30)

Let $F^s_{p,q}(Q)$ (can be extended to cellular domains) be given with

$$1 \le p < \infty, 0 < q < \infty, s > \sigma_q, s > \frac{1}{p}, u > s \text{ and } s - \frac{k}{p} \notin \mathbb{N}_0$$

for $k = 1, \ldots, n$.

Then there is an oscillating u-wavelet system which is a Riesz basis in $F^s_{p,q}(Q)$.

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Then there is an oscillating u-wavelet system which is a Riesz basis in $F^s_{p,q}(Q)$.

Proof: Use the decomposition

$$F^s_{p,q}(Q) = \tilde{F}^s_{p,q}(Q) \times Ext_\Gamma^{\bar{r},u} \prod_{\ell=\ell_0}^{n-1} \prod_{j=0}^{N_\ell} \prod_{|\alpha| \le r^\ell} \tilde{F}_{p,p}^{s-\frac{n-\ell}{p}-|\alpha|}(Q_{\ell,j})$$

and the fact that every space on the right hand side has a u-wavelet system which is a Riesz basis. Here we need $s > \sigma_q$ for the wavelet basis of $\tilde{F}^s_{p,q}(Q)$.

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The model case $s = \frac{1}{p}$ for C^∞-domains Ω

Let $1 < p < \infty$, $1 \leq q < \infty$ and let Ω be a C^∞-domain. Then

$$\{f \in F^{\frac{1}{p}}_{p,q}(\Omega) : d^{-1/p} f \in L_p(\Omega)\} = \tilde{F}^{\frac{1}{p}}_{p,q}(\Omega) \subsetneq \mathring{F}^{\frac{1}{p}}_{p,q}(\Omega) = F^{\frac{1}{p}}_{p,q}(\Omega),$$

where $d(x) = \mathrm{dist}(x, \partial\Omega)$.
Example: Take χ_Ω, characteristic function of Ω.

The model case $s = \frac{1}{p}$ for C^∞-domains Ω

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where $d(x) = \mathrm{dist}(x, \partial\Omega)$.
Example: Take χ_Ω, characteristic function of Ω.
Altogether, we have

- $\frac{1}{p} - 1 < s < \frac{1}{p}$: $\tilde{F}^s_{p,q}(\Omega) = \mathring{F}^s_{p,q}(\Omega) = F^s_{p,q}(\Omega)$
- $s = \frac{1}{p}$: $\tilde{F}^s_{p,q}(\Omega) \subsetneq \mathring{F}^s_{p,q}(\Omega) = F^s_{p,q}(\Omega)$, no interior u-Riesz frames for $\mathring{F}^s_{p,q}(\Omega)$ (otherwise $\mathring{F}^s_{p,q}(\Omega) = \tilde{F}^s_{p,q}(\Omega)$, [T08], Prop. 6.40)
- $\frac{1}{p} < s < \frac{1}{p} + 1$: $\tilde{F}^s_{p,q}(\Omega) = \mathring{F}^s_{p,q}(\Omega) = \{f \in F^s_{p,q}(\Omega) : \mathrm{tr}_{\partial\Omega} f = 0\}$.

Reinforcing $F^s_{p,q}(\Omega)$ for C^∞-domains Ω

Prof. Triebel: "If the mountain does not come to the prophet, the prophet has to come to the mountain!"

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Definition

Let $1 \leq p < \infty$, $0 < q \leq \infty$ and $s > \frac{1}{p} - 1$. Let Ω be a C^∞-domain.
(i) Let $s - \frac{1}{p} \notin \mathbb{N}_0$. Then

$$F^{s,\mathrm{rinf}}_{p,q}(\Omega) := F^s_{p,q}(\Omega).$$

(ii) Let $s - \frac{1}{p} = r \in \mathbb{N}_0$. Then

$$F^{s,\mathrm{rinf}}_{p,q}(\Omega) := \left\{ f \in F^s_{p,q}(\mathbb{R}^n) : d^{-\frac{1}{p}} \cdot \frac{\partial^r f}{\partial \nu^r} \in L_p(\Omega_\varepsilon) \right\}.$$

Wavelet frames for C^∞-domains Ω (i)

Theorem (T08 - Theorem 6.46)

Let Ω be a C^∞-domain. Let $u > s$ and

$$1 \leq p < \infty, 1 \leq q < \infty, s > \frac{1}{p} - 1.$$

Then there is an oscillating u-wavelet system which is a Riesz frame in $F^{s,\mathrm{rinf}}_{p,q}(\Omega)$.

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Proof: Of interest are the cases $s - \frac{1}{p} = r \in \mathbb{N}_0$. One shows

$$\tilde{F}^s_{p,q}(\Omega) = \{f \in F^{s,\mathrm{rinf}}_{p,q}(\Omega) : tr^{r-1}_\Gamma f = 0\}.$$

This is the substitute for

$$\tilde{F}^s_{p,q}(\Omega) = \{f \in F^s_{p,q}(\Omega) : tr^r_\Gamma f = 0\},$$

which holds for $s - \frac{1}{p} \notin \mathbb{N}_0$.

Wavelet frames for C^∞-domains Ω (ii)

Furthermore, we have

$$Ext^{r-1,u} : \prod_{|\alpha|\leq r-1} \tilde{F}^{s-\frac{1}{p}-|\alpha|}_{p,p}(\Gamma) \mapsto F^{s,\mathrm{rinf}}_{p,q}(\Omega)$$

instead of

$$Ext^{r,u} : \prod_{|\alpha|\leq r} \tilde{F}^{s-\frac{1}{p}-|\alpha|}_{p,p}(\Gamma) \mapsto F^{s}_{p,q}(\Omega)$$

by construction of the (wavelet-friendly) extension operator. Hence

$$f = \underbrace{f - Ext^{r-1,u} \circ tr^{r-1}_{\Gamma}}_{\in\{f\in F^{s,\mathrm{rinf}}_{p,q}(\Omega): tr^{r-1}_{\Gamma} f=0\}} + \underbrace{Ext^{r-1,u} \circ tr^{r-1}_{\Gamma}}_{\in Ext \prod_{i=1}^{r-1} \tilde{F}^{s-\frac{1}{p}-|\alpha|}_{p,p}(\Gamma)} .$$

The situation for cubes Q in the exceptional cases

Roughly speaking:

- a C^∞-domain Ω has only boundaries of dimension $n-1 \to$ exceptional values for $s-\frac{1}{p} \in \mathbb{N}_0$
- the cube Q has boundaries of dimension 0 to $n-1 \to$ exceptional values for $s-\frac{k}{p} \in \mathbb{N}_0$ for $k=1,\ldots,n$

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Example (Grisvard '85, '92)

The space $H^1(Q) = F^1_{2,2}(Q)$ is exceptional: $s - \frac{2}{p} = 2 - 1$. Let $\Gamma = \partial\Omega = I_1 \cup I_2 \cup I_3 \cup I_4$. Then the trace space $tr_\Gamma H^1(Q)$ is the collection of all tuples $g = (g_1, g_2, g_3, g_4)$ with

$$g_\ell \in H^{\frac{1}{2}}(I_\ell), \quad \ell = 1, 2, 3, 4$$

and

$$\int_0^{1/2} \frac{|g_1(t) - g_2(t)|^2}{t} \, dt < \infty, \text{ etc.}$$

At first: The domains $\mathbb{R}^n \setminus \mathbb{R}^\ell$

Definition

Let $n \in \mathbb{N}$ with $\ell < n$. Let $\mathbb{R}^n = \mathbb{R}^\ell \times \mathbb{R}^{n-\ell}$ and $x = (y, z) \in \mathbb{R}^n$,

$$y = (y_1, \ldots, y_\ell) \in \mathbb{R}^\ell, z = (z_1, \ldots, z_{n-\ell}) \in \mathbb{R}^{n-\ell}.$$

We identify $\mathbb{R}^\ell$ with the hyperplane $\{z = 0\} \subset \mathbb{R}^n$. Hence, in our understanding

$$\mathbb{R}^n \setminus \mathbb{R}^\ell = \{x = (y, z) \in \mathbb{R}^n : z \neq 0\}.$$

Let

$$d(x) = \operatorname{dist}(x, \mathbb{R}^\ell) = \inf\{|x - y| : y \in \mathbb{R}^\ell\} = |z|,$$

$$\text{and} \quad \Omega_\varepsilon = \{x \in \Omega : d(x) < \varepsilon\}, \varepsilon << 1$$

Reinforced spaces for $\mathbb{R}^n \setminus \mathbb{R}^\ell$ (i)

Definition

Let $1 \leq p < \infty$, $0 < q \leq \infty$ and $s > \frac{n-\ell}{p} - 1$.

(i) Let $s - \frac{n-\ell}{p} \notin \mathbb{N}_0$. Then

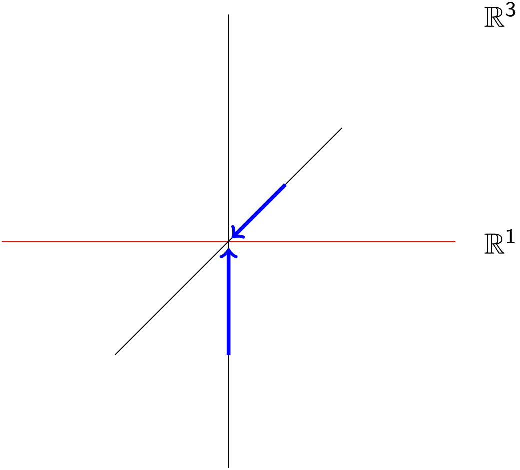
$$F^{s,\mathrm{rinf}}_{p,q}(\mathbb{R}^n \setminus \mathbb{R}^\ell) := F^s_{p,q}(\mathbb{R}^n).$$

(ii) Let $s - \frac{n-\ell}{p} = r \in \mathbb{N}_0$. Then

$$F^{s,\mathrm{rinf}}_{p,q}(\mathbb{R}^n \setminus \mathbb{R}^\ell) := \left\{ f \in F^s_{p,q}(\mathbb{R}^n) : d^{-\frac{n-\ell}{p}} \cdot D^\alpha f \in L_p((\mathbb{R}^n \setminus \mathbb{R}^\ell)_\varepsilon) \ \forall \alpha \in \mathbb{N}^n_\ell : |\alpha| = r \right\}.$$

This is the generalization of $F^{s,\mathrm{rinf}}_{p,q}(\Omega)$ for an ℓ-dimensional boundary. Set $\ell = n - 1$ and this looks like before - then only 1 derivative!

Reinforced spaces for $\mathbb{R}^n \setminus \mathbb{R}^\ell$ (ii)



The substitute for $\tilde{F}^s_{p,q}(\Omega)$ on $\mathbb{R}^n \setminus \mathbb{R}^\ell$

It holds

$$\tilde{F}^s_{p,q}(\mathbb{R}^n \setminus \mathbb{R}^\ell) \cong F^s_{p,q}(\mathbb{R}^n) \text{ since } \bar{\Omega} = \mathbb{R}^n \text{ and } \tilde{F}^s_{p,q}(\partial\Omega) = \{0\}$$

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We need a proper substitute!

Proposition (T08 - Prop. 3.10)

Let Ω be an Lipschitz (E-thick) domain in $\mathbb{R}^n$. Let

$$0 < p < \infty, 0 < q < \infty, s > \sigma_{p,q}.$$

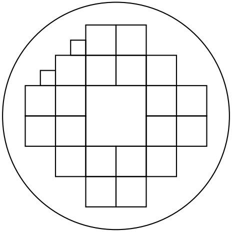
Then

$$F^{s,\mathrm{rloc}}_{p,q}(\Omega) = \tilde{F}^s_{p,q}(\Omega).$$

What is $F^{s,\mathrm{rloc}}_{p,q}(\Omega)$?

Whitney decomposition of Ω

Let Ω be an arbitrary open domain and let $Q^0_{j,r}$, $Q^1_{j,r}$ be the Whitney cubes:



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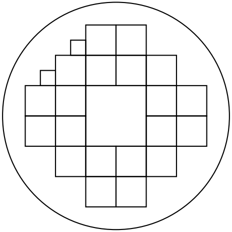

Let $Q^0_{\ell,r}, Q^1_{\ell,r}$ be centred at $2^{-\ell}m$ with $m^r \in \mathbb{Z}^n$ and side-length $2^{-\ell+1}$ resp. $2^{-\ell+2}$ such that

$$Q_{\ell,r} \text{ disjoint}, \Omega = \bigcup_{\ell,r} \overline{Q^0_{\ell,r}} \text{ and } \operatorname{dist}(Q^1_{\ell,r}, \Gamma) \sim 2^{-\ell} \text{ if } \ell \in \mathbb{N}$$

Further $\operatorname{dist}(Q^1_{0,r}, \Gamma) \geq c$ and $|\ell - \ell'| \leq 1$ for two adjacent cubes.

Refined localization spaces of Ω

Let $Q^0_{\ell,r}, Q^1_{\ell,r}$ be the Whitney cubes of Ω. Let $\varrho = \{\varrho_{j,r}\}$ be a suitable resolution of unity, i. e.

$$\mathit{supp}\ \varrho_{j,r} \subset Q^1_{j,r}, \quad \|D^\alpha \varrho_{j,r}(x)\| \le c_\alpha 2^{j|\alpha|}, x \in \Omega, \alpha \in \mathbb{N}_0^n$$

for some $c_\alpha > 0$ independent of x, j, r and

$$\sum_{j=0}^{\infty} \sum_r \varrho_{j,r}(x) = 1 \text{ if } x \in \Omega.$$

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Definition

Let Ω be an open domain. Let $0 \le p < \infty$, $0 < q \le \infty$, $s > \sigma_{p,q}$. Then

$$F^{s,\mathrm{rloc}}_{p,q}(\Omega) := \left\{ f \in D'(\Omega) : \|f|F^{s,\mathrm{rloc}}_{p,q}(\Omega)\|_\rho < \infty \right\}$$

with

$$\|f|F^{s,\mathrm{rloc}}_{p,q}(\Omega)\|^p_\rho = \sum_{j=0}^{\infty} \sum_r \|\varrho_{j,r} f|F^s_{p,q}(\mathbb{R}^n)\|^p.$$

Wavelet bases for refined localization spaces

The following theorem gives an alternative approach to define $F^{s,\mathrm{rloc}}_{p,q}(\Omega)$ which is nowadays maybe the more common way.

Theorem

Let Ω be an arbitrary domain in $\mathbb{R}^n$ with $\Omega \neq \mathbb{R}^n$. Let

$$0 < p < \infty, 0 < q < \infty, s > \sigma_{p,q} \text{ and } u > s.$$

Then there is an orthonormal u-wavelet basis

$$\Phi = \left\{ \Phi^j_r : j \in \mathbb{N}_0, r = 1, \ldots, N_j \right\} \in C^u(\Omega)$$

in $L_2(\Omega)$ which is an interior u-Riesz basis for the function space $F^{s,\mathrm{rloc}}_{p,q}(\Omega)$ with the sequence space $f^s_{p,q}(\mathbb{Z}_\Omega)$. It holds

$$\lambda^j_r(f) = 2^{jn/2}(f, \Phi^j_r).$$

The interplay of $F^{s,\mathrm{rloc}}_{p,q}(\Omega)$ and $F^{s,\mathrm{rinf}}_{p,q}(\Omega)$, $\Omega = \mathbb{R}^n \setminus \mathbb{R}^\ell$ (i)

Theorem (follows from T08, Theorem 2.18)

Let Ω be an arbitrary domain in $\mathbb{R}^n$ with $\Omega \neq \mathbb{R}^n$, let

$$1 \leq p < \infty, 0 < q < \infty, s > \sigma_q.$$

Then $f \in F^{s,\mathrm{rloc}}_{p,q}(\Omega)$ if, and only if,

$$\|f|F^s_{p,q}(\Omega)\| + \|d^{-s} \cdot f|L_p(\Omega_\varepsilon)\| < \infty$$

(equivalent norms) with $d(x) = \mathrm{dist}(x, \partial\Omega)$.

Hence for $\sigma_q < s \leq \frac{n-\ell}{p}$ we have

$$F^{s,\mathrm{rinf}}_{p,q}(\mathbb{R}^n \setminus \mathbb{R}^\ell) = F^{s,\mathrm{rloc}}_{p,q}(\mathbb{R}^n \setminus \mathbb{R}^\ell).$$

The interplay of $F^{s,\mathrm{rloc}}_{p,q}(\Omega)$ and $F^{s,\mathrm{rinf}}_{p,q}(\Omega)$, $\Omega = \mathbb{R}^n \setminus \mathbb{R}^\ell$ (ii)

Theorem (Decomposition)

Let $1 \leq p < \infty$ *and* $0 < q \leq \infty$. *Let* $n \in \mathbb{N}$, $\ell, r \in \mathbb{N}_0$ *or* $r = -1$ *with* $\ell < n$,

$$s > \sigma_q$$

and

$$r = \left\lceil s - \frac{n-\ell}{p} \right\rceil - 1.$$

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Then

$$F^{s,\mathrm{rloc}}_{p,q}(\mathbb{R}^n \setminus \mathbb{R}^\ell) = \left\{ f \in F^{s,\mathrm{rinf}}_{p,q}(\mathbb{R}^n \setminus \mathbb{R}^\ell) : \mathrm{tr}^r_\ell f = 0 \right\}.$$

with no condition if $r = -1 \Leftrightarrow \sigma_q < s \le \frac{n-\ell}{p}$.

Reinforced spaces for cubes Q

Let $\Gamma_{\ell,j}$ be the ℓ-dimensional faces of the cube Q. With $\mathbb{N}^n_{\ell,j}$ we denote the multi-indices with directions perpendicular to $\Gamma_{\ell,j}$.

Definition

We say that $f \in F^s_{p,q}(\mathbb{R}^n)$ has the reinforced property $R^{r,p}_\ell$ if, and only if,

$$d^{-\frac{n-\ell}{p}} \cdot D^\alpha f \in L_p((\mathbb{R}^n \setminus \Gamma_{\ell,j})_\varepsilon)$$

for all $\alpha \in \mathbb{N}^n_\ell, |\alpha| = r$ and $j = 1, \dots, n_\ell$.

Reinforced spaces for cubes Q (ii)

Definition

Let $1 \leq p < \infty$, $0 < q < \infty$, $s > \sigma_q$. Let

$$F^{s,\mathrm{rinf}}_{p,q}(Q) := F^{s,\mathrm{rinf}}_{p,q}(\mathbb{R}^n \setminus \Gamma)_{|Q}$$

with usual inf-norm and

$$F^{s,\mathrm{rinf}}_{p,q}(\mathbb{R}^n \setminus \Gamma) :=$$

$$\{f \in F^s_{p,q}(\mathbb{R}^n) : \forall\, 0 \leq \ell \leq n-1 : f \text{ fulfils } R^{r,p}_{\ell} \text{ if } r = k - \frac{n-\ell}{p} \in \mathbb{N}_0\}$$

Wavelet bases for reinforced function spaces on cubes Q

Theorem (main goal - not yet carried out)

Let

$$1 < p < \infty, 1 \leq q < \infty, s > 0, u > s.$$

Then there is an oscillating u-wavelet system which is a Riesz basis in $F^{s,\mathrm{rinf}}_{p,q}(Q)$*.*

This is a generalization of the wavelet theorem for $s - \frac{k}{p} \notin \mathbb{N}_0$.

An example - $H^{1,\text{rinf}}(Q) = F^{1,\text{rinf}}_{2,2}(Q)$ for $n = 2$ (i)

We have $s - \frac{2}{p} = r = 0$ - faces of dimension 0 are problematic. Hence

$$H^{1,\text{rinf}}(Q) = F^{1,\text{rinf}}_{2,2}(Q) = \{f \in F^1_{2,2}(Q) : \int_Q |f(x)|^2 \frac{dx}{d(x)^2}\},$$

where d is the distance from the corner points of Q.

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where d is the distance from the corner points of Q. Let

$$f = (f - Ext_0 \circ tr_0) + Ext_0 \circ tr_0 =: f_1 + f_2.$$

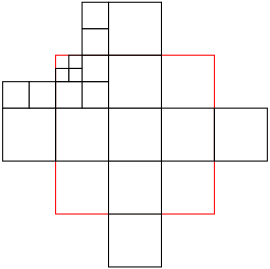
Then $tr_0 f_1 = 0$. Hence by our decomposition theorem for an extension $\tilde{f}_1$ of f_1 to $\mathbb{R}^n$

$$\tilde{f}_1 \in F^{s,\text{rloc}}_{p,q}(\mathbb{R}^n \setminus \Gamma_0)$$

and so has $\tilde{f}_1$ a wavelet decomposition avoiding the corner points:

An example - $H^{1,\mathrm{rinf}}(Q) = F^{1,\mathrm{rinf}}_{2,2}(Q)$ for $n = 2$ (i)

The interesting fact: If $\tilde{f}_1$, then $\mathrm{tr}_1\, \tilde{f}_1$ has a wavelet decomposition on Γ_1. Hence $\mathrm{tr}_1\, \tilde{f}_1 \in F^{s,\mathrm{rloc}}_{p,q}(\Gamma_1)$.



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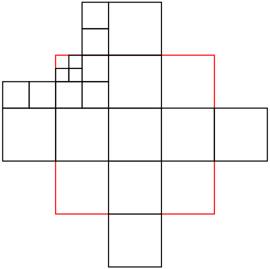

Altogether:

$$\begin{aligned} f = f_1 + f_2 &= f_1 - (Ext_1 \circ tr_1)f_1 + (Ext_1 \circ tr_1)f_1 + f_2 \\ &\in \{F^{s,\mathrm{rinf}}_{p,q}(Q) : tr_1 f = 0\} + Ext_1 F^{s,\mathrm{rloc}}_{p,q}(\Gamma_1) + Ext_2 F^{s,\mathrm{rloc}}_{p,q}(\Gamma_0) \\ &= \text{``}F^{s,\mathrm{rloc}}_{p,q}(\mathbb{R}^n \setminus \Gamma_2)\text{``} + Ext_1 \tilde{F}^s_{p,q}(\Gamma_1) + Ext_2 \tilde{F}^s_{p,q}(\Gamma_0). \end{aligned}$$

The end

Thank you for your attention

Questions?