Wavelets in function spaces on cellular domains

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Domains in $\mathbb{R}^n$ (i)

(i) Let $2 \leq n$. A special Lipschitz ($C^\infty$-) domain in $\mathbb{R}^n$ is the collection of all points $x = (x', x_n)$ with $x' \in \mathbb{R}^{n-1}$ and $h(x') < x_n$, where $h$ is a Lipschitz (bounded $C^\infty$-) function.
Domains in $\mathbb{R}^n$ (i)

(i) Let $2 \leq n$. A special Lipschitz ($C^\infty$-) domain in $\mathbb{R}^n$ is the collection of all points $x = (x', x_n)$ with $x' \in \mathbb{R}^{n-1}$ and $h(x') < x_n$, where $h$ is a Lipschitz (bounded $C^\infty$-) function.

(ii) Let $2 \leq n$. A bounded Lipschitz ($C^\infty$-) domain is a bounded domain $\Omega$ where the boundary $\Gamma = \partial \Omega$ can be covered by finitely many open balls $B_j$ centred at $\Gamma$ such that $B_j \cap \Omega = B_j \cap \Omega_j$, where $\Omega_j$ are rotations of suitable special Lipschitz ($C^\infty$-) domains.
Domains in $\mathbb{R}^n$ (i)

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(ii) Let $2 \leq n$. A bounded Lipschitz ($C^\infty$-) domain is a bounded domain $\Omega$ where the boundary $\Gamma = \partial \Omega$ can be covered by finitely many open balls $B_j$ centred at $\Gamma$ such that $B_j \cap \Omega = B_j \cap \Omega_j$, where $\Omega_j$ are rotations of suitable special Lipschitz ($C^\infty$-) domains.

(iii) Let $2 \leq n$. A domain $\Omega$ in $\mathbb{R}^n$ is called cellular if it is a bounded Lipschitz domain which can be represented as

$$\Omega = \left( \bigcup_{\ell=1}^L \bar{\Omega}_\ell \right)^\circ \quad \text{with} \quad \Omega_\ell \cap \Omega_{\ell'} = \emptyset \text{ if } \ell \neq \ell'$$

such that each $\Omega_\ell$ is diffeomorphic to a polyhedron (cube).
Domains in $\mathbb{R}^n$ (ii)
The definition of $B_{p,q}^s(\mathbb{R}^n)$

Let $\{\varphi_j\}_{j=0}^\infty$ be a resolution of unity. Let $0 < p \leq \infty$, $0 < q \leq \infty$ and $s \in \mathbb{R}$. For $f \in S'(\mathbb{R}^n)$ we define

$$\|f|B_{p,q}^s(\mathbb{R}^n)|_\varphi := \left( \sum_{j=0}^\infty 2^{jsq} \| (\varphi_j \hat{f})^\gamma \|_{L_p(\mathbb{R}^n)} \right)^{\frac{1}{q}}$$

(modified in case $q = \infty$) and

$$B_{p,q}^{s,\varphi}(\mathbb{R}^n) := \{ f \in S'(\mathbb{R}^n) : \|f|B_{p,q}^s(\mathbb{R}^n)|_\varphi < \infty \}.$$ 

Then $(B_{p,q}^{s,\varphi}(\mathbb{R}^n), \| \cdot |B_{p,q}^s(\mathbb{R}^n)|_\varphi)$ is a quasi-Banach space. It does not depend on the choice of the resolution of unity $\{\varphi_j\}_{j=0}^\infty$ in the sense of equivalent norms. So we denote it shortly by $B_{p,q}^s(\mathbb{R}^n)$. 
The definition of $F_{p,q}^s(\mathbb{R}^n)$

Let $\{\varphi_j\}_{j=0}^\infty$ be a resolution of unity. Let $0 < p < \infty$, $0 < q \leq \infty$ and $s \in \mathbb{R}$. For $f \in S'(\mathbb{R}^n)$ we define

$$\|f|F_{p,q}^s(\mathbb{R}^n)\|_{\varphi} := \left\| \sum_{j=0}^{\infty} 2^{jsq} |(\varphi_j \hat{f})|^q \right\|_{L_p(\mathbb{R}^n)}^{\frac{1}{q}}$$

(modified in case $q = \infty$) and

$$F_{p,q}^{s,\varphi}(\mathbb{R}^n) := \{ f \in S'(\mathbb{R}^n) : \|f|F_{p,q}^s(\mathbb{R}^n)\|_{\varphi} < \infty \} .$$

Then $(F_{p,q}^{s,\varphi}(\mathbb{R}^n), \| \cdot |F_{p,q}^s(\mathbb{R}^n)\|_{\varphi})$ is a quasi-Banach space. It does not depend on the choice of the resolution of unity $\{\varphi_j\}_{j=0}^\infty$ in the sense of equivalent norms. So we denote it shortly by $F_{p,q}^s(\mathbb{R}^n)$. 
The definition of $A_{p,q}^s(\Omega)$, $\tilde{A}_{p,q}^s(\Omega)$ and $\hat{A}_{p,q}^s(\Omega)$ (i)

Let $\Omega$ be an open set in $\mathbb{R}^n$. Denote by $g|\Omega \in D'(\Omega)$ its restriction to $\Omega$, hence $(g|\Omega)(\varphi) = g(\varphi)$ for $\varphi \in D(\Omega)$.

$$A_{p,q}^s(\Omega) := \{ f \in D'(\Omega) : f = g|\Omega \text{ for some } g \in A_{p,q}^s(\mathbb{R}^n) \},$$

$$\| f|A_{p,q}^s(\Omega) \| = \inf \| g|A_{p,q}^s(\mathbb{R}^n) \|,$$

where the infimum is taken over all $g \in A_{p,q}^s(\mathbb{R}^n)$ with $g|\Omega = f$. 
The definition of \( A^s_{p,q}(\Omega), \tilde{A}^s_{p,q}(\Omega) \) and \( \breve{A}^s_{p,q}(\Omega) \) (i)

Let \( \Omega \) be an open set in \( \mathbb{R}^n \). Denote by \( g|\Omega \in D'(\Omega) \) its restriction to \( \Omega \), hence \((g|\Omega)(\varphi) = g(\varphi)\) for \( \varphi \in D(\Omega) \).

\[
A^s_{p,q}(\Omega) := \{ f \in D'(\Omega) : f = g|\Omega \text{ for some } g \in A^s_{p,q}(\mathbb{R}^n) \},
\]
\[
\| f|A^s_{p,q}(\Omega) \| = \inf \| g|A^s_{p,q}(\mathbb{R}^n) \|,
\]

where the infimum is taken over all \( g \in A^s_{p,q}(\mathbb{R}^n) \) with \( g|\Omega = f \).

Moreover, let

\[
\tilde{A}^s_{p,q}(\overline{\Omega}) := \{ f \in A^s_{p,q}(\mathbb{R}^n) : \text{supp } f \in \overline{\Omega} \}
\]

with the quasi-norm from \( A^s_{p,q}(\mathbb{R}^n) \). Then

\[
\breve{A}^s_{p,q}(\Omega) := \{ f \in D'(\Omega) : f = g|\Omega \text{ for some } g \in \tilde{A}^s_{p,q}(\overline{\Omega}) \},
\]
\[
\| f|\breve{A}^s_{p,q}(\Omega) \| = \inf \| g|\tilde{A}^s_{p,q}(\overline{\Omega}) \|
\]

where the infimum is taken over all \( g \in \tilde{A}^s_{p,q}(\overline{\Omega}) \) with \( g|\Omega = f \).
The definition of $A^s_{p,q}(\Omega)$, $\tilde{A}^s_{p,q}(\Omega)$ and $\hat{A}^s_{p,q}(\Omega)$ (ii)

It holds

$$A^s_{p,q}(\Omega) = A^s_{p,q}(\mathbb{R}^n) / \tilde{A}^s_{p,q}(\Omega^C),$$

$$\tilde{A}^s_{p,q}(\Omega) = \tilde{A}^s_{p,q}((\tilde{\Omega}) / \tilde{A}^s_{p,q}(\partial \Omega))$$

in the sense of quotient spaces and norms. Hence $A^s_{p,q}(\Omega)$ and $\tilde{A}^s_{p,q}(\Omega)$ are quasi-Banach spaces.
The definition of $A_{p,q}^s(\Omega)$, $\tilde{A}_{p,q}^s(\Omega)$ and $\check{A}_{p,q}^s(\Omega)$ (ii)

It holds

$$A_{p,q}^s(\Omega) = A_{p,q}^s(\mathbb{R}^n)/\check{A}_{p,q}^s(\Omega^c),$$

$$\tilde{A}_{p,q}^s(\Omega) = \tilde{A}_{p,q}^s(\bar{\Omega})/\check{A}_{p,q}^s(\partial\Omega)$$

in the sense of quotient spaces and norms. Hence $A_{p,q}^s(\Omega)$ and $\tilde{A}_{p,q}^s(\Omega)$ are quasi-Banach spaces.

Let $\Omega$ be a cellular domain and let

$$0 < p \leq \infty, 0 < q \leq \infty, \max\left(n \left(\frac{1}{p} - 1\right), \frac{1}{p} - 1\right) < s$$

($p < \infty$ for the $F$-spaces). Then by T08, Prop. 6.13.

$$\tilde{A}_{p,q}^s(\Omega) = \tilde{A}_{p,q}^s(\bar{\Omega})$$

in the sense of $\tilde{A}_{p,q}^s(\partial\Omega) = \{0\}$. 
The definition of $A^s_{p,q}(\Omega)$, $\tilde{A}^s_{p,q}(\Omega)$ and $\hat{A}^s_{p,q}(\Omega)$ (iii)
The definition of $A_{p,q}^s(\Omega)$, $\tilde{A}_{p,q}^s(\Omega)$ and $\hat{A}_{p,q}^s(\Omega)$ (iii)

Furthermore, let $\hat{A}_{p,q}^s(\Omega)$ be the completion of $D(\Omega)$ with respect to $\| \cdot |A_{p,q}^s(\Omega)\|$.

**Theorem (The starting stripe - T08, Prop. 6.13.)**

Let $\Omega$ be a cellular domain and let

$$0 < p < \infty, 0 < q < \infty, \max \left( n \left( \frac{1}{p} - 1 \right), \frac{1}{p} - 1 \right) < s < \frac{1}{p}$$

Then

$$A_{p,q}^s(\Omega) = \hat{A}_{p,q}^s(\Omega) = \tilde{A}_{p,q}^s(\Omega).$$
The definition of $A_{p,q}^s(\Omega)$, $\tilde{A}_{p,q}^s(\Omega)$ and $\hat{A}_{p,q}^s(\Omega)$ (iii)

Furthermore, let $\hat{A}_{p,q}^s(\Omega)$ be the completion of $D(\Omega)$ with respect to $\| \cdot \|_{A_{p,q}^s(\Omega)}$.

**Theorem (The starting stripe - T08, Prop. 6.13.)**

Let $\Omega$ be a cellular domain and let

$$0 < p < \infty, 0 < q < \infty, \max \left( n \left( \frac{1}{p} - 1 \right), \frac{1}{p} - 1 \right) < s < \frac{1}{p}$$

Then

$$A_{p,q}^s(\Omega) = \hat{A}_{p,q}^s(\Omega) = \tilde{A}_{p,q}^s(\Omega).$$

Proof: Use that $\chi_{\Omega}$ is a pointwise multiplier in these spaces.
The starting stripe
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3 The exceptional values of $s$
   • Some known results
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Wavelets for function spaces on domains $\Omega$ (i)

From now on let $\Omega$ be a cellular domain and $\Gamma = \partial \Omega$. Let

$$\mathbb{Z}^\Omega := \left\{ x^j_\ell \in \Omega : j \in \mathbb{N}_0, \ell = 1, \ldots, N_j \right\},$$

(typically $N_j \sim 2^{jn}$), such that for some $c_1 > 0$

$$|x^j_\ell - x^j_{\ell'}| \geq c_1 2^{-j}, \ j \in \mathbb{N}_0, \ \ell \neq \ell'.$$
Wavelets for function spaces on domains \( \Omega \) (i)

From now on let \( \Omega \) be a cellular domain and \( \Gamma = \partial \Omega \). Let

\[
\mathbb{Z}^\Omega := \left\{ x^j_\ell \in \Omega : j \in \mathbb{N}_0, \ell = 1, \ldots, N_j \right\},
\]

(typically \( N_j \sim 2^{jn} \)), such that for some \( c_1 > 0 \)

\[
|x^j_\ell - x^{j\prime}_\ell'| \geq c_1 2^{-j}, \quad j \in \mathbb{N}_0, \ell \neq \ell'.
\]

We can introduce the usual sequence spaces \( b^s_{p,q}(\mathbb{Z}^\Omega) \) and \( f^s_{p,q}(\mathbb{Z}^\Omega) \) adapted to \( \mathbb{Z}^\Omega \) and abbreviated by \( a^s_{p,q}(\mathbb{Z}^\Omega) \). For example,

\[
\| \lambda |f^s_{p,q}(\mathbb{Z}^\Omega)| \| := \left\| \left( \sum_{j=0}^{\infty} \sum_{\ell=1}^{N_j} 2^{jsq} |x^j_\ell \cdot \chi_{j,\ell}|^q(\cdot) \right)^\frac{1}{q} \|_{L^p(\Omega)} \right\|
\]

where \( \chi_{j,\ell} \) is the characteristic function of \( B(x^j_\ell, c2^{-j}) \) or \( \bar{\Omega} \cap B(x^j_\ell, c2^{-j}) \) with an arbitrary constant \( c > 0 \).
Wavelets for function spaces on domains $\Omega$ (ii)

Let $u \in \mathbb{N}_0$.

Then

$$\Phi = \left\{ \Phi^j_\ell : j \in \mathbb{N}_0, \ell = 1, \ldots, N_j \right\} \subset C^u(\Omega)$$

is called a $u$-wavelet system in $\bar{\Omega}$ (adapted to $\mathbb{Z}^\Omega$) if it fulfils

- **support conditions**: For some $c_3 > 0$ it holds

  $$\text{supp} \Phi^j_\ell \subset B(x^j_\ell, c_3 2^{-j}) \cap \bar{\Omega}, \ j \in \mathbb{N}_0, \ \ell = 1, \ldots, N_j,$$
Wavelets for function spaces on domains $\Omega$ (ii)

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- **support conditions**: For some $c_3 > 0$ it holds

  $$\text{supp} \, \Phi^j_\ell \subset B(x^j_\ell, c_3 2^{-j}) \cap \bar{\Omega}, \quad j \in \mathbb{N}_0, \ell = 1, \ldots, N_j,$$

- **derivative conditions**: For some $c_4 > 0$ and all $\alpha \in \mathbb{N}_0^n$ with

  $$0 \leq |\alpha| \leq u$$

  let

  $$\left| D^\alpha \Phi^j_\ell(x) \right| \leq c_4 2^j \frac{n}{2} + j |\alpha|, \quad x \in \Omega, j \in \mathbb{N}_0, \ell = 1, \ldots, N_j.$$
Additionally, the $u$-wavelet system is called oscillating if it fulfils

- (substitute) moment conditions: Let $c_5$ and $c_6 < c_7$ be constants such that

\[
\text{dist}(B(x^0_\ell, c_3), \Gamma) \geq c_6, \text{ for } \ell = 1, \ldots, \mathbb{N}_0 \text{ and }
\]

\[
\left| \int_{\Omega} \psi(x) \Phi^j_\ell(x) \, dx \right| \leq c_5 2^{-j/2} - j^u \| \psi \| C^u(\Omega) \text{ for all } \psi \in C^u(\Omega)
\]

for all $j \in \mathbb{N}$ and $\ell$ with $\text{dist}(B(x^j_\ell, c_3), \Gamma) \notin (c_6 2^{-j}, c_7 2^{-j})$. 
Additionally, the $u$-wavelet system is called oscillating if it fulfils

- (substitute) moment conditions: Let $c_5$ and $c_6 < c_7$ be constants such that

\[ \text{dist}(B(x_0^\ell, c_3), \Gamma) \geq c_6, \text{ for } \ell = 1, \ldots, N_0 \text{ and } \]
\[ \left| \int_{\Omega} \psi(x) \Phi^j_\ell(x) \, dx \right| \leq c_5 2^{-j} 2^{-j} \| \psi \|_{C^u(\Omega)} \text{ for all } \psi \in C^u(\Omega) \]

for all $j \in \mathbb{N}$ and $\ell$ with $\text{dist}(B(x^j_\ell, c_3), \Gamma) \notin (c_6 2^{-j}, c_7 2^{-j})$.

An oscillating $u$-wavelet system is called interior if it fulfils

- (further) interior support conditions, namely

\[ \text{dist}(B(x^j_\ell, c_3 2^{-j}), \Gamma) \geq c_6 2^{-j}, j \in \mathbb{N}_0, \ell = 1, \ldots, N_j. \]
Wavelet bases in $L_2(\Omega)$ - the starting point

**Theorem (T08 - Theorem 2.33)**

Let $\Omega$ be an arbitrary domain in $\mathbb{R}^n$. For any $u \in \mathbb{N}_0$ there is a

$$\Phi = \left\{ \Phi_j^\ell : j \in \mathbb{N}_0, \ell = 1, \ldots, N_j \right\} \subset C^u(\Omega)$$

which is

1. an orthonormal basis in $L_2(\Omega)$,
2. an interior $u$-wavelet system

simultaneously.
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1. an orthonormal basis in $L_2(\Omega)$,
2. an interior $u$-wavelet system

simultaneously.

The wavelets even fulfil “true” moment conditions

$$\int_\Omega x^\alpha \Phi^j_\ell(x) \, dx = 0 \text{ for } |\alpha| \leq u$$

inside the domain.

For $u = 0$ one can take the Haar Wavelet suitably restricted.
Wavelet bases in $\tilde{A}^s_{p,q}(\Omega)$ and $L_p(\Omega)$

**Theorem (T08 - Theorem 2.36, Prop. 3.10)**

Let $\Omega$ be a Lipschitz ($E$-thick) domain in $\mathbb{R}^n$. Let $u > s > \sigma_p$ resp. $> \sigma_{p,q}$. Then the interior $u$-wavelet system orthonormal in $L_2(\Omega)$ is a Riesz basis for $L_p(\Omega) = \tilde{F}^0_{p,2}(\Omega), 1 < p < \infty$, for $\tilde{B}^s_{p,q}(\Omega)$ and $\tilde{F}^s_{p,q}(\Omega)$:

For $f \in L^1_{\text{loc}}$ (resp. $D'(\Omega)$) it holds

\[
f \in \tilde{A}^s_{p,q}(\Omega) \iff f = \sum_{j=0}^{\infty} \sum_{r=1}^{N_j} \lambda^j_r 2^{-jn/2} \Phi^j_r, \lambda \in a^s_{p,q}(\mathbb{Z}_\Omega).
\]

Moreover, the representation is unique with $\lambda^j_r(f) = 2^{jn/2} (f, \Phi^j_r)$ and

\[
\|f|\tilde{A}^s_{p,q}(\Omega)\| \sim \|\lambda|a^s_{p,q}(\mathbb{Z}_\Omega)\|.
\]
Wavelet bases in $A_{p,q}^s(\Omega)$

**Theorem (T08 - Prop. 3.13)**

Let $\Omega$ be a Lipschitz ($E$-thick) domain in $\mathbb{R}^n$. Let $s < 0$ and $u > \sigma_p - s$ resp. $u > \sigma_{p,q} - s$. Then the interior $u$-wavelet system orthonormal in $L_2(\Omega)$ is a Riesz basis for $B_{p,q}^s(\Omega)$ and $F_{p,q}^s(\Omega)$: For $f \in D'(\Omega)$ it holds

$$f \in A_{p,q}^s(\Omega) \Leftrightarrow f = \sum_{j=0}^{\infty} \sum_{r=1}^{N_j} \lambda^j_r 2^{-jn/2} \Phi^j_r, \lambda \in a_{p,q}^s(\mathbb{Z} \Omega).$$

Moreover, the representation is unique with $\lambda^j_r(f) = 2^{jn/2} (f, \Phi^j_r)$ and

$$\|f| A_{p,q}^s(\Omega)\| \sim \|\lambda| a_{p,q}^s(\mathbb{Z} \Omega)\|. $$
Wavelet bases in $A^s_{p,q}(\Omega)$

**Theorem (T08 - Prop. 3.13)**

Let $\Omega$ be a Lipschitz ($E$-thick) domain in $\mathbb{R}^n$. Let $s < 0$ and $u > \sigma_p - s$ resp. $u > \sigma_{p,q} - s$. Then the interior $u$-wavelet system orthonormal in $L_2(\Omega)$ is a Riesz basis for $B^s_{p,q}(\Omega)$ and $F^s_{p,q}(\Omega)$: For $f \in D'(\Omega)$ it holds

$$ f \in A^s_{p,q}(\Omega) \iff f = \sum_{j=0}^{\infty} \sum_{r=1}^{N_j} \lambda_r 2^{-jn/2} \Phi_j^r, \lambda \in a^s_{p,q}(\mathbb{Z}_\Omega). $$

Moreover, the representation is unique with $\lambda^i_r(f) = 2^{jn/2} (f, \Phi^i_r)$ and

$$ \| f | A^s_{p,q}(\Omega) \| \sim \| \lambda | a^s_{p,q}(\mathbb{Z}_\Omega) \|. $$

Proof (main idea) of the theorems: Wavelets serve as atoms and local means. To generate moment conditions for atoms ($s < 0$) or local means ($s > 0$) one has to extend the boundary wavelets. For $s > 0$ hence $f$ needs support in $\tilde{\Omega}$. 
Corollary (Homogeneity property - T08, Theorem 2.11.)

Let $U_\lambda$ be either the balls or cubes with radius resp. diameter $\lambda$. Then for $s > \sigma_p$ resp. $s > \sigma_{p,q}$

$$\|f(\lambda \cdot) | \tilde{A}_{p,q}^s(U_1) \| \sim \lambda^{s - \frac{n}{p}} \| f | \tilde{A}_{p,q}^s(U_\lambda) \|.$$ 

For $s < 0$ it holds

$$\|f(\lambda \cdot) | A_{p,q}^s(U_1) \| \sim \lambda^{s - \frac{n}{p}} \| f | A_{p,q}^s(U_\lambda) \|,$$

where the equivalence constants in both cases are independent of $0 < \lambda \leq 1$ and of $f$. 

Wavelet bases in $A_{p,q}^s(\Omega)$, $\tilde{A}_{p,q}^s(\Omega)$ and $L_p$
Wavelet bases in $A^s_{p,q}(\Omega)$, $\tilde{A}^s_{p,q}(\Omega)$ and $L_p$

Corollary (Homogeneity property - T08, Theorem 2.11.)

Let $U_\lambda$ be either the balls or cubes with radius resp. diameter $\lambda$. Then for $s > \sigma_p$ resp. $s > \sigma_{p,q}$

$$\| f(\lambda \cdot) |\tilde{\Lambda}^s_{p,q}(U_1)\| \sim \lambda^{s - \frac{n}{p}} \| f |\tilde{\Lambda}^s_{p,q}(U_\lambda)\|.$$  

For $s < 0$ it holds

$$\| f(\lambda \cdot) |A^s_{p,q}(U_1)\| \sim \lambda^{s - \frac{n}{p}} \| f |A^s_{p,q}(U_\lambda)\|,$$

where the equivalence constants in both cases are independent of $0 < \lambda \leq 1$ and of $f$.

Proof idea: The wavelet representations of $U_{2^{-j}}$ and $U_1$ can be transformed into each other by dilation with $2^{-j}$.
Wavelet bases in the starting stripe

Corollary (T10 - Theorem 2.13., T08 - Prop. 3.21.)

Let $\Omega$ be a cellular domain and let $1 \leq p < \infty$, $0 < q < \infty$ resp. $1 \leq q < \infty$. Then for $u \geq 1$

$$\Phi = \left\{ \Phi_j^\ell : j \in \mathbb{N}_0, \ell = 1, \ldots, N_j \right\} \subset C^u(\Omega)$$

(and the Haar system) are Riesz bases for $A_{p,q}^s(\Omega)$ in the starting stripe

$$\frac{1}{p} - 1 < s < \frac{1}{p}.$$
Wavelet bases in the starting stripe

Corollary (T10 - Theorem 2.13., T08 - Prop. 3.21.)

Let $\Omega$ be a cellular domain and let $1 \leq p < \infty$, $0 < q < \infty$ resp. $1 \leq q < \infty$. Then for $u \geq 1$

$$\Phi = \left\{ \Phi_j^\ell : j \in \mathbb{N}_0, \ell = 1, \ldots, N_j \right\} \subset C^u(\Omega)$$

(and the Haar system) are Riesz bases for $A_{s_p,q}^s(\Omega)$ in the starting stripe

$$\frac{1}{p} - 1 < s < \frac{1}{p}.$$

Proof idea: By $A_{p,q}^s(\Omega) = \tilde{A}_{p,q}^s(\Omega)$ the cases with $s \neq 0$ follow from the theorems before. The rest is a matter of interpolation.

**Note:** For $s > \frac{1}{p}$ there are no **interior** $u$-wavelet systems for $A_{p,q}^s(\Omega)$ because of boundary values. We have to find “non-interior” wavelet systems.
Traces on the boundary of cubes (i)

Let $Q = \{ x \in \mathbb{R}^n : x = (x_1, \ldots, x_n), 0 < x_m < 1, m = 1, \ldots, n \}$. The boundary $\Gamma = \partial Q$ of $Q$ can be represented as

$$\Gamma = \bigcup_{\ell=0}^{n-1} \Gamma_\ell \text{ with } \Gamma_\ell \cap \Gamma_{\ell'} = \emptyset \text{ for } \ell \neq \ell',$$

where $\Gamma_\ell = \bigcup_{j=0}^{N_\ell} Q_{\ell,j}$ consists of all $\ell$-dimensional faces $Q_{\ell,j}$ of $Q$, which are disjoint cubes of dimension $\ell$. 
Traces on the boundary of cubes (i)

Let $Q = \{x \in \mathbb{R}^n : x = (x_1, \ldots, x_n), 0 < x_m < 1, m = 1, \ldots, n\}$. The boundary $\Gamma = \partial Q$ of $Q$ can be represented as

$$\Gamma = \bigcup_{\ell=0}^{n-1} \Gamma_{\ell} \text{ with } \Gamma_{\ell} \cap \Gamma_{\ell'} = \emptyset \text{ for } \ell \neq \ell',$$

where $\Gamma_{\ell} = \bigcup_{j=0}^{N_\ell} Q_{\ell,j}$ consists of all $\ell$-dimensional faces $Q_{\ell,j}$ of $Q$, which are disjoint cubes of dimension $\ell$. 

![Diagram showing traces on the boundary of cubes](image)
Traces on the boundary of cubes (ii)

Let \( tr_{\ell,j} \) be the restriction of \( f \in A^s_{p,q}(\mathbb{R}^n) \) to \( Q_{\ell,j} \) and

\[
tr^r_{\ell} : f \mapsto TR^r_{\ell}(f) := \prod \{ tr_{\ell,j} D_\gamma^\alpha f : |\alpha| \leq r, j = 0, \ldots, N_\ell \},
\]

where only derivatives perpendicular to \( Q_{\ell,j} \) are admitted.
Traces on the boundary of cubes (ii)

Let \( tr_{\ell,j} \) be the restriction of \( f \in A_s^{p,q}(\mathbb{R}^n) \) to \( Q_{\ell,j} \) and

\[
tr^r_{\ell} : f \mapsto TR^r_{\ell}(f) := \prod \left\{ tr_{\ell,j} D_\gamma^\alpha f : |\alpha| \leq r, j = 0, \ldots, N_{\ell} \right\},
\]

where only derivatives perpendicular to \( Q_{\ell,j} \) are admitted. Then we consider the composite mapping for \( \vec{r} = (r^{\ell_0}, \ldots, r^{n-1}) \), \( \ell_0 \leq n - 1 \) and \( r^{\ell} = \lfloor s - \frac{n-\ell}{p} \rfloor \):

\[
tr^\vec{r} f : f \mapsto \prod_{\ell = \ell_0}^{n-1} TR^{\vec{r}}_\ell (f).
\]

\( Q_{0,j} : 3 \) directions, \( 1 + \lfloor s - \frac{3}{p} \rfloor \) derivatives

\( Q_{1,j} : 2 \) directions, \( 1 + \lfloor s - \frac{2}{p} \rfloor \) derivatives

\( Q_{2,j} : 1 \) direction, \( 1 + \lfloor s - \frac{1}{p} \rfloor \) derivatives
Traces on the boundary of cubes (iii)

**Theorem**

Let

\[ 1 \leq p < \infty, 0 < q < \infty, s > \frac{1}{p} \text{ and } s - \frac{k}{p} \notin \mathbb{N}_0 \text{ for } k = 1, \ldots, n. \]

Let \( \ell_0 = 0 \) if \( s > \frac{n}{p} \). Otherwise \( \ell_0 \in \mathbb{N} \) is chosen such that

\[ 0 < s - \frac{n - \ell_0}{p} < \frac{1}{p}. \]

Then

\[ \text{tr}^\Gamma : B^s_{p,q}(Q) \mapsto \prod_{\ell=\ell_0}^{n-1} \prod_{j=0}^{N_\ell} \prod_{|\alpha| \leq r^\ell} B^{s-\frac{n-\ell}{p}-|\alpha|}_{p,q}(Q_{\ell,j}), \]

\[ \text{tr}^\Gamma : F^s_{p,q}(Q) \mapsto \prod_{\ell=\ell_0}^{n-1} \prod_{j=0}^{N_\ell} \prod_{|\alpha| \leq r^\ell} B^{s-\frac{n-\ell}{p}-|\alpha|}_{p,p}(Q_{\ell,j}). \]
Traces on the boundary of cubes (iv)

Theorem

Let

\[ 1 \leq p < \infty, 0 < q < \infty, s > \frac{1}{p} \quad \text{and} \quad s - \frac{k}{p} \notin \mathbb{N}_0 \text{ for } k = 1, \ldots, n. \]

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\[ 0 < s - \frac{n - \ell_0}{p} < \frac{1}{p}. \]

Then

\[ \tilde{B}^s_{p,q}(Q) = \{ f \in B^s_{p,q}(Q) : \text{tr} \tilde{r}_\Gamma = 0 \}, \]

\[ \tilde{F}^s_{p,q}(Q) = \{ f \in F^s_{p,q}(Q) : \text{tr} \tilde{r}_\Gamma = 0 \}. \]
Extension operators for the boundary of cubes (i)

**Theorem**

Let $p, q, s, \ell_0$ as in the theorem before and additionally $s < u$. Then there is a **wavelet-friendly** extension operator

$$
\text{Ext}^{\tilde{r}, u}_\Gamma : \prod_{\ell = \ell_0}^{n-1} N_\ell \prod_{j = 0}^{N_\ell} \prod_{|\alpha| \leq r_\ell} \tilde{B}_{p,q}^{s - \frac{n-\ell}{p} - |\alpha|} (Q_{\ell,j}) \mapsto B_{p,q}^s (Q),
$$

$$
\text{Ext}^{\tilde{r}, u}_\Gamma : \prod_{\ell = \ell_0}^{n-1} N_\ell \prod_{j = 0}^{N_\ell} \prod_{|\alpha| \leq r_\ell} \tilde{B}_{p,p}^{s - \frac{n-\ell}{p} - |\alpha|} (Q_{\ell,j}) \mapsto F_{p,q}^s (Q).
$$

It holds

$$
\text{tr}_\Gamma \circ \text{ext}^{\tilde{r}, u}_\Gamma = \text{id}.
$$
Extension operators for the boundary of cubes (ii)

Theorem

Furthermore,

\[ B_{p,q}^s(Q) = \tilde{B}_{p,q}^s(Q) \times \text{Ext}_{\Gamma}^{\tilde{r},u} \prod_{\ell=\ell_0}^{n-1} \prod_{j=0}^{N_\ell} \prod_{|\alpha| \leq r^\ell} \tilde{B}_{p,q}^{s-n-\ell} - |\alpha| (Q_{\ell,j}), \]

\[ F_{p,q}^s(Q) = \tilde{F}_{p,q}^s(Q) \times \text{Ext}_{\Gamma}^{\tilde{r},u} \prod_{\ell=\ell_0}^{n-1} \prod_{j=0}^{N_\ell} \prod_{|\alpha| \leq r^\ell} \tilde{B}_{p,p}^{s-n-\ell} - |\alpha| (Q_{\ell,j}). \]
Extension operators for the boundary of cubes (ii)

**Theorem**

Furthermore,

\[
B^s_{p,q}(Q) = \tilde{B}^s_{p,q}(Q) \times \text{Ext}_{\Gamma}^{\bar{r},u} \prod_{\ell=\ell_0}^{n-1} \prod_{j=0}^{N_\ell} \prod_{|\alpha| \leq r^\ell} \tilde{B}^{s-n-\ell}_{p,p} - |\alpha| (Q_{\ell,j}),
\]

\[
F^s_{p,q}(Q) = \tilde{F}^s_{p,q}(Q) \times \text{Ext}_{\Gamma}^{\bar{r},u} \prod_{\ell=\ell_0}^{n-1} \prod_{j=0}^{N_\ell} \prod_{|\alpha| \leq r^\ell} \tilde{B}^{s-n-\ell}_{p,p} - |\alpha| (Q_{\ell,j}).
\]

**Proof:** Use

\[
\tilde{B}^s_{p,q}(Q) = \{ f \in B^s_{p,q}(Q) : tr_{\Gamma}^\bar{r} = 0 \},
\]

\[
\tilde{F}^s_{p,q}(Q) = \{ f \in F^s_{p,q}(Q) : tr_{\Gamma}^\bar{r} = 0 \}.
\]
Wavelets for cubes

Theorem (T08 - Theorem 6.30)

Let $A^s_{p,q}(Q)$ (can be extended to cellular domains) be given with

$$1 \leq p < \infty, \ s > \frac{1}{p} \ \text{and} \ s - \frac{k}{p} \notin \mathbb{N}_0 \ \text{for} \ k = 1, \ldots, n, 0 < q < \infty$$

($q \geq 1$ for the $F$-spaces). Then there is an oscillating $u$-wavelet system with $u > s$ which is a Riesz basis in $A^s_{p,q}(Q)$. 
Wavelets for cubes

**Theorem (T08 - Theorem 6.30)**

Let $A^s_{p,q}(Q)$ (can be extended to cellular domains) be given with

\[ 1 \leq p < \infty, s > \frac{1}{p} \text{ and } s - \frac{k}{p} \notin \mathbb{N}_0 \text{ for } k = 1, \ldots, n, 0 < q < \infty \]

$q \geq 1$ for the $F$-spaces). Then there is an oscillating $u$-wavelet system with $u > s$ which is a Riesz basis in $A^s_{p,q}(Q)$.

Proof: Use the decomposition

\[
B^s_{p,q}(Q) = \tilde{B}^s_{p,q}(Q) \times \text{Ext}^{\vec{r},u} \prod_{\ell = \ell_0}^{n-1} \prod_{j=0}^{N_{\ell}} \prod_{|\alpha| \leq r^{\ell}} \tilde{B}^s_{p,q} - \frac{n-\ell}{p} - |\alpha| (Q_{\ell,j})
\]

and the fact that every space on the right hand side has a $u$-wavelet system which is a Riesz basis. Now one must ensure that wavelet-friendly extension operators are really wavelet-friendly.
The case $s - \frac{1}{p} \in \mathbb{N}_0$

Let $\Omega$ now be a $C^\infty$-domain (so also a cellular domain) and let $1 < p, q < \infty$. By a translation and localization argument one always has

$$\tilde{\mathcal{A}}_{s,p,q}(\Omega) \hookrightarrow \mathcal{A}_{s,p,q}(\Omega).$$

The converse is true if $s - \frac{1}{p} \notin \mathbb{N}$. Using Hardy inequalities from [T01] one gets

$$\int_{\Omega} \frac{|f(x)|^p}{d^{sp}(x)} \, dx \leq c \|f| F_{s,p,q}(\mathbb{R}^n)\| \text{ for } f \in \tilde{\mathcal{F}}_{s,p,q}(\Omega),$$

where $d(x) = \text{dist}(x, \Gamma)$. Then $\chi_\Omega \notin \tilde{\mathcal{F}}_{s,p,q}(\Omega)$ for $s \geq \frac{1}{p}$. 
The case \( s - \frac{1}{p} \in \mathbb{N}_0 \)

Let \( \Omega \) now be a \( C^\infty \)-domain (so also a cellular domain) and let \( 1 < p, q < \infty \). By a translation and localization argument one always has

\[
\tilde{A}_{p,q}^s(\Omega) \hookrightarrow \dot{A}_{p,q}^s(\Omega).
\]

The converse is true if \( s - \frac{1}{p} \notin \mathbb{N} \). Using Hardy inequalities from [T01] one gets

\[
\int_{\Omega} \frac{|f(x)|^p}{d^{sp}(x)} \, dx \leq c \| f \|_{F_{p,q}^s(\mathbb{R}^n)} \quad \text{for } f \in \tilde{F}_{p,q}^s(\Omega),
\]

where \( d(x) = dist(x, \Gamma) \). Then \( \chi_\Omega \notin \tilde{F}_{p,q}^s(\Omega) \) for \( s \geq \frac{1}{p} \).

On the other hand \( \chi_\Omega \in \dot{F}_{p,q}^{\frac{1}{p}}(\Omega) = F_{p,q}^{\frac{1}{p}}(\Omega) \) by an atomic decomposition argument.

Analogously, it follows \( \tilde{F}_{p,q}^s(\Omega) \neq \dot{F}_{p,q}^s(\Omega) \) if \( s - \frac{1}{p} \in \mathbb{N} \). There are no Riesz frames for these \( \tilde{F}_{p,q}^s(\Omega) \) (T08 - Prop. 6.40.).
The cases $s - \frac{k}{p} \in \mathbb{N}_0$ with $k = 2, \ldots, n$

It is known that not all the conditions are necessary depending on the kind of domain.

- For planar $C^\infty$-domains ($n = 2$) the restriction for $k = 2$ is not necessary,
- For $C^\infty$-domains with boundaries diffeomorphic to $\mathbb{S}^n$ the condition for $k = n$ is not necessary,
- For arbitrary $C^\infty$-domains and
  
  $1 \leq p < \infty, \frac{1}{p} - 1 < s < \frac{2}{p}, s \neq \frac{1}{p}$

one finds a Riesz u-wavelet basis without further restrictions since the boundary has a basis,

- For the torus one needs no further conditions since the boundary is equipped with a u-wavelet basis ([T08] - sect. 1.3.2). So $W_2^1(\Omega)$ has a basis if $\Omega$ is a torus which is unknown if $\Omega$ is a ball and $n \geq 3$. 
Reinforced spaces

Let $\Omega_\varepsilon = \{ x \in \Omega, d(x) < \varepsilon \}$ and

$$F_{p,q}^{s,r_{inf}}(\Omega) = \begin{cases} F_{p,q}^{s}(\Omega) & \text{if } s - \frac{1}{p} \notin \mathbb{N}_0 \\ f \in F_{p,q}^{s}(\Omega) : d^{-\frac{1}{p}} \frac{\partial f}{\partial \nu} \in L_p(\Omega_\varepsilon) & \text{if } s - \frac{1}{p} = r \in \mathbb{N}_0. \end{cases}$$

If $\Omega$ is a bounded $C^\infty$-domain, then

$$\tilde{F}_{p,q}^{s}(\Omega) = \left\{ f \in F_{p,q}^{s,r_{inf}}(\Omega) : \text{tr}_{r^{-1}} f = 0 \right\}.$$
Reinforced spaces

Let $\Omega_\varepsilon = \{ x \in \Omega, d(x) < \varepsilon \}$ and

$$F^{s,\text{rinf}}_{p,q}(\Omega) = \begin{cases} F^s_{p,q}(\Omega) & \text{if } s - \frac{1}{p} \notin \mathbb{N}_0 \\ f \in F^s_{p,q}(\Omega) : d^{-\frac{1}{p}} \frac{\partial f}{\partial \nu} \in L^p(\Omega_\varepsilon) & \text{if } s - \frac{1}{p} = r \in \mathbb{N}_0. \end{cases}$$

If $\Omega$ is a bounded $C^\infty$-domain, then

$$\tilde{F}^s_{p,q}(\Omega) = \left\{ f \in F^{s,\text{rinf}}_{p,q}(\Omega) : tr^{r-1}_f f = 0 \right\}.$$

This and the observation that the known extension operators also map to $F^{s,\text{rinf}}_{p,q}(\Omega)$ instead of $F^s_{p,q}(\Omega)$ is the starting point for finding a $u$-wavelet basis.
Reinforced spaces advanced

Roughly speaking: This gives u-wavelet bases for the modified spaces $F_{p,q}^{s,r_{\text{inf}}}(\Omega)$ in the cases of domains where no more restrictions than $s - \frac{1}{p} \notin \mathbb{N}_0$ are necessary.
Reinforced spaces advanced

Roughly speaking: This gives u-wavelet bases for the modified spaces $F_{s,r_{inf}}^{p,q}(\Omega)$ in the cases of domains where no more restrictions than $s - \frac{1}{p} \notin \mathbb{N}_0$ are necessary.

The big question: Can one extend this idea to cellular domains with more restrictions?
Reinforced spaces advanced

Roughly speaking: This gives $u$-wavelet bases for the modified spaces $F_{p,q}^{s,r\text{inf}}(\Omega)$ in the cases of domains where no more restrictions than $s - \frac{1}{p} \notin \mathbb{N}_0$ are necessary.

The big question: Can one extend this idea to cellular domains with more restrictions?

The problem: One has to ensure a composition of type

$$\tilde{F}_{p,q}^{s}(\Omega) = \left\{ f \in F_{p,q}^{s,r\text{inf}}(\Omega) : tr_{r-1}f = 0 \right\}.$$

and has to check if the extension operators are suitable (which they should be). Furthermore, we need conditions for defining $F_{p,q}^{s,r\text{inf}}(\Omega)$ which handle the lower dimensional boundary faces. This has not been done so far. In [T08], Sect. 6.2.4, there is some overview given for $W^1_2(Q)$. 
The end

Thank you for your attention

Questions?