Wavelets in function spaces on cellular domains

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Domains in $\mathbb{R}^n$ (i)

(i) Let $2 \leq n$. A special Lipschitz (C^∞-) domain in $\mathbb{R}^n$ is the collection of all points $x = (x', x_n)$ with $x' \in \mathbb{R}^{n-1}$ and $h(x') < x_n$, where h is a Lipschitz (bounded C^∞-) function.

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(ii) Let $2 \leq n$. A bounded Lipschitz (C^∞-) domain is a bounded domain Ω where the boundary $\Gamma = \partial\Omega$ can be covered by finitely many open balls B_j centred at Γ such that $B_j \cap \Omega = B_j \cap \Omega_j$, where Ω_j are rotations of suitable special Lipschitz (C^∞-) domains.

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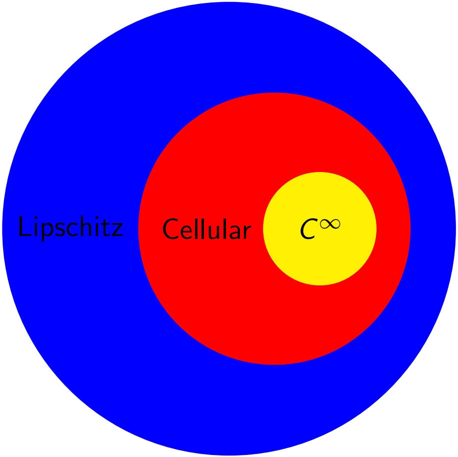
(iii) Let $2 \leq n$. A domain Ω in $\mathbb{R}^n$ is called cellular if it is a bounded Lipschitz domain which can be represented as

$$\Omega = \left(\bigcup_{\ell=1}^{L} \bar{\Omega}_\ell \right)^\circ \text{ with } \Omega_\ell \cap \Omega_{\ell'} = \emptyset \text{ if } \ell \neq \ell'$$

such that each Ω_ℓ is diffeomorphic to a polyhedron (cube).

Domains in $\mathbb{R}^n$ (ii)



The definition of $B^s_{p,q}(\mathbb{R}^n)$

Let $\{\varphi_j\}_{j=0}^{\infty}$ be a resolution of unity. Let $0 < p \leq \infty$, $0 < q \leq \infty$ and $s \in \mathbb{R}$. For $f \in \mathcal{S}'(\mathbb{R}^n)$ we define

$$\|f|B^s_{p,q}(\mathbb{R}^n)\|^{\varphi} := \left(\sum_{j=0}^{\infty} 2^{jsq} \|(\varphi_j \hat{f})^{\vee}|L_p(\mathbb{R}^n)\|^q \right)^{\frac{1}{q}}$$

(modified in case $q = \infty$) and

$$B^{s,\varphi}_{p,q}(\mathbb{R}^n) := \left\{ f \in \mathcal{S}'(\mathbb{R}^n) : \|f|B^s_{p,q}(\mathbb{R}^n)\|^{\varphi} < \infty \right\}.$$

Then $(B^{s,\varphi}_{p,q}(\mathbb{R}^n), \|\cdot|B^s_{p,q}(\mathbb{R}^n)\|^{\varphi})$ is a quasi-Banach space. It does not depend on the choice of the resolution of unity $\{\varphi_j\}_{j=0}^{\infty}$ in the sense of equivalent norms. So we denote it shortly by $B^s_{p,q}(\mathbb{R}^n)$.

The definition of $F^s_{p,q}(\mathbb{R}^n)$

Let $\{\varphi_j\}_{j=0}^\infty$ be a resolution of unity. Let $0 < p < \infty$, $0 < q \le \infty$ and $s \in \mathbb{R}$. For $f \in \mathcal{S}'(\mathbb{R}^n)$ we define

$$\|f|F^s_{p,q}(\mathbb{R}^n)\|^\varphi := \left\| \left(\sum_{j=0}^{\infty} 2^{jsq} |(\varphi_j \hat{f})^\vee|^q \right)^{\frac{1}{q}} \Big| L_p(\mathbb{R}^n) \right\|$$

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The definition of $A^s_{p,q}(\Omega)$, $\tilde{A}^s_{p,q}(\Omega)$ and $\mathring{A}^s_{p,q}(\Omega)$ (i)

Let Ω be an open set in $\mathbb{R}^n$. Denote by $g|\Omega \in D'(\Omega)$ its restriction to Ω, hence $(g|\Omega)(\varphi) = g(\varphi)$ for $\varphi \in D(\Omega)$.

$$A^s_{p,q}(\Omega) := \{f \in D'(\Omega) : f = g|\Omega \text{ for some } g \in A^s_{p,q}(\mathbb{R}^n)\},$$
$$\|f|A^s_{p,q}(\Omega)\| = \inf \|g|A^s_{p,q}(\mathbb{R}^n)\|,$$

where the infimum is taken over all $g \in A^s_{p,q}(\mathbb{R}^n)$ with $g|\Omega = f$.

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where the infimum is taken over all $g \in A^s_{p,q}(\mathbb{R}^n)$ with $g|\Omega = f$.
Moreover, let

$$\tilde{A}^s_{p,q}(\bar{\Omega}) := \{f \in A^s_{p,q}(\mathbb{R}^n) : \mathit{supp}\, f \in \bar{\Omega}\}$$

with the quasi-norm from $A^s_{p,q}(\mathbb{R}^n)$. Then

$$\tilde{A}^s_{p,q}(\Omega) := \{f \in D'(\Omega) : f = g|\Omega \text{ for some } g \in \tilde{A}^s_{p,q}(\bar{\Omega})\},$$
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The definition of $A^s_{p,q}(\Omega)$, $\tilde{A}^s_{p,q}(\Omega)$ and $\mathring{A}^s_{p,q}(\Omega)$ (ii)

It holds

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\begin{aligned}
A^s_{p,q}(\Omega) &= A^s_{p,q}(\mathbb{R}^n)/\tilde{A}^s_{p,q}(\Omega^C),\\
\tilde{A}^s_{p,q}(\Omega) &= \tilde{A}^s_{p,q}(\bar{\Omega})/\tilde{A}^s_{p,q}(\partial\Omega)
\end{aligned}
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in the sense of quotient spaces and norms. Hence $A^s_{p,q}(\Omega)$ and $\tilde{A}^s_{p,q}(\Omega)$ are quasi-Banach spaces.

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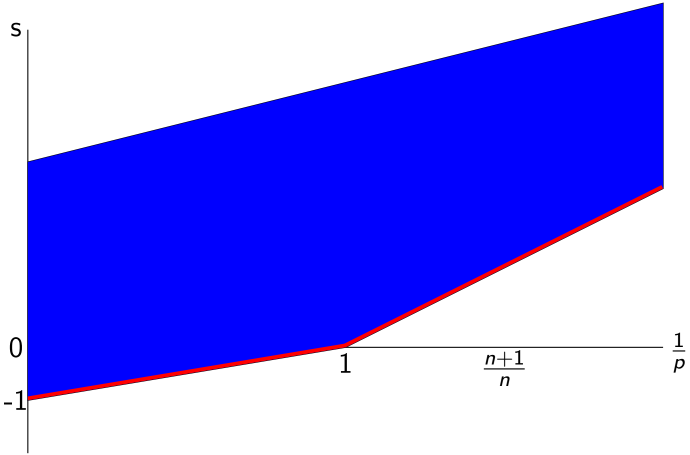
Let Ω be a cellular domain and let

$$0 < p \leq \infty, 0 < q \leq \infty, \max\left(n\left(\frac{1}{p}-1\right), \frac{1}{p}-1\right) < s$$

($p < \infty$ for the F-spaces). Then by T08, Prop. 6.13.

$$\tilde{A}^s_{p,q}(\Omega) = \tilde{A}^s_{p,q}(\bar{\Omega}) \text{ in the sense of } \tilde{A}^s_{p,q}(\partial\Omega) = \{0\}.$$

The definition of $A^s_{p,q}(\Omega)$, $\tilde{A}^s_{p,q}(\Omega)$ and $\mathring{A}^s_{p,q}(\Omega)$ (iii)



The definition of $A^s_{p,q}(\Omega)$, $\tilde{A}^s_{p,q}(\Omega)$ and $\mathring{A}^s_{p,q}(\Omega)$ (iii)

Furthermore, let $\mathring{A}^s_{p,q}(\Omega)$ be the completion of $D(\Omega)$ with respect to $\| \cdot |A^s_{p,q}(\Omega)\|$.

Theorem (The starting stripe - T08, Prop. 6.13.)

Let Ω be a cellular domain and let

$$0 < p < \infty, 0 < q < \infty, \max\left(n\left(\frac{1}{p}-1\right), \frac{1}{p}-1\right) < s < \frac{1}{p}$$

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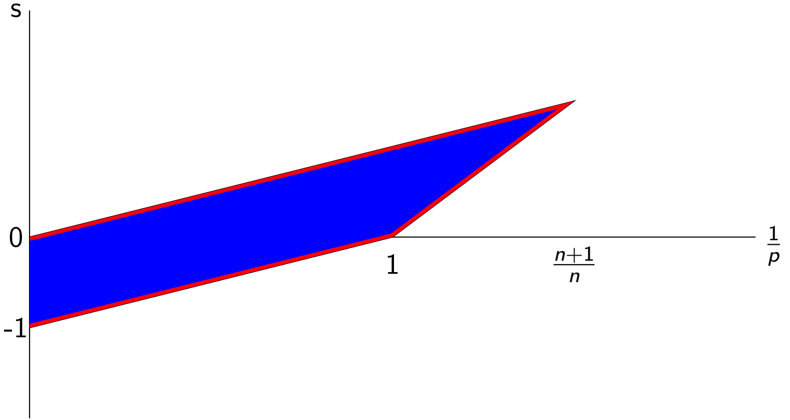
Then

$$A^s_{p,q}(\Omega) = \mathring{A}^s_{p,q}(\Omega) = \tilde{A}^s_{p,q}(\Omega).$$

Proof: Use that χ_Ω is a pointwise multiplier in these spaces.

The starting stripe



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Wavelets for function spaces on domains Ω (i)

From now on let Ω be a cellular domain and $\Gamma = \partial\Omega$. Let

$$\mathbb{Z}^\Omega := \left\{ x_\ell^j \in \Omega : j \in \mathbb{N}_0, \ell = 1, \ldots, N_j \right\},$$

(typically $N_j \sim 2^{jn}$), such that for some $c_1 > 0$

$$|x_\ell^j - x_{\ell'}^j| \geq c_1 2^{-j},\ j \in \mathbb{N}_0,\ \ell \neq \ell'.$$

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$$|x_\ell^j - x_{\ell'}^j| \geq c_1 2^{-j}, \; j \in \mathbb{N}_0, \; \ell \neq \ell'.$$

We can introduce the usual sequence spaces $b^s_{p,q}(\mathbb{Z}^\Omega)$ and $f^s_{p,q}(\mathbb{Z}^\Omega)$ adapted to $\mathbb{Z}^\Omega$ and abbreviated by $a^s_{p,q}(\mathbb{Z}^\Omega)$. For example,

$$\|\lambda|f^s_{p,q}(\mathbb{Z}^\Omega)\| := \left\| \left(\sum_{j=0}^{\infty} \sum_{\ell=1}^{N_j} 2^{jsq} |\lambda_\ell^j \cdot \chi_{j,\ell}|^q(\cdot) \right)^{\frac{1}{q}} |L_p(\Omega) \right\|,$$

where $\chi_{j,\ell}$ is the characteristic function of $B(x_\ell^j, c2^{-j})$ or $\bar{\Omega} \cap B(x_\ell^j, c2^{-j})$ with an arbitrary constant $c > 0$.

Wavelets for function spaces on domains Ω (ii)

Let $u \in \mathbb{N}_0$.
Then

$$\Phi = \left\{ \Phi_\ell^j : j \in \mathbb{N}_0, \ell = 1, \dots, N_j \right\} \subset C^u(\Omega)$$

is called a u-wavelet system in $\bar{\Omega}$ (adapted to $\mathbb{Z}^\Omega$) if it fulfils

- support conditions: For some $c_3 > 0$ it holds

$$supp\, \Phi_\ell^j \subset B(x_\ell^j, c_3 2^{-j}) \cap \bar{\Omega},\ j \in \mathbb{N}_0,\ \ell = 1, \dots, N_j,$$

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- derivative conditions: For some $c_4 > 0$ and all $\alpha \in \mathbb{N}^n_0$ with $0 \le |\alpha| \le u$ let

$$\left| D^\alpha \Phi^j_\ell(x) \right| \le c_4 2^{j\frac{n}{2} + j|\alpha|}, x \in \Omega\, j \in \mathbb{N}_0,\ \ell = 1, \dots, N_j.$$

Wavelets for function spaces on domains Ω (iii)

Additionally, the u-wavelet system is called oscillating if it fulfils

- (substitute) moment conditions: Let c_5 and $c_6 < c_7$ be constants such that

$$dist(B(x_\ell^0, c_3), \Gamma) \geq c_6, \text{ for } \ell = 1, \dots, \mathbb{N}_0 \text{ and}$$
$$\left| \int_\Omega \psi(x) \Phi_\ell^j(x)\, dx \right| \leq c_5 2^{-j\frac{n}{2} - ju} \|\psi | C^u(\Omega)\| \text{ for all } \psi \in C^u(\Omega)$$

for all $j \in \mathbb{N}$ and ℓ with $dist(B(x_\ell^j, c_3), \Gamma) \notin \left(c_6 2^{-j}, c_7 2^{-j}\right)$.

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$$\left| \int_\Omega \psi(x) \Phi_\ell^j(x) \, dx \right| \leq c_5 2^{-j\frac{n}{2} - ju} \left\| \psi | C^u(\Omega) \right\| \text{ for all } \psi \in C^u(\Omega)$$

for all $j \in \mathbb{N}$ and ℓ with $dist(B(x_\ell^j, c_3), \Gamma) \notin \left(c_6 2^{-j}, c_7 2^{-j}\right)$.

An oscillating u-wavelet system is called interior if it fulfils

- (further) interior support conditions, namely

$$dist(B(x_\ell^j, c_3 2^{-j}), \Gamma) \geq c_6 2^{-j}, \, j \in \mathbb{N}_0, \, \ell = 1, \dots, N_j.$$

Wavelet bases in $L_2(\Omega)$ - the starting point

Theorem (T08 - Theorem 2.33)

Let Ω be an arbitrary domain in $\mathbb{R}^n$. For any $u \in \mathbb{N}_0$ there is a

$$\Phi = \left\{ \Phi^j_\ell : j \in \mathbb{N}_0, \ell = 1, \ldots, N_j \right\} \subset C^u(\Omega)$$

which is

1. *an orthonormal basis in $L_2(\Omega)$,*
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simultaneously.

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The wavelets even fulfil "true" moment conditions

$$\int_\Omega x^\alpha \Phi_\ell^j(x) \, dx = 0 \text{ for } |\alpha| \le u$$

inside the domain.

For $u = 0$ one can take the Haar Wavelet suitably restricted.

Wavelet bases in $\tilde{A}^s_{p,q}(\Omega)$ and $L_p(\Omega)$

Theorem (T08 - Theorem 2.36, Prop. 3.10)

Let Ω be a Lipschitz (E-thick) domain in $\mathbb{R}^n$. Let $u > s > \sigma_p$ resp. $> \sigma_{p,q}$. Then the interior u-wavelet system orthonormal in $L_2(\Omega)$ is a Riesz basis for $L_p(\Omega) = \tilde{F}^0_{p,2}(\bar{\Omega})$, $1 < p < \infty$, for $\tilde{B}^s_{p,q}(\Omega)$ and $\tilde{F}^s_{p,q}(\Omega)$: For $f \in L_1^{loc}$ (resp. $D'(\Omega)$) it holds

$$f \in \tilde{A}^s_{p,q}(\Omega) \Leftrightarrow f = \sum_{j=0}^{\infty} \sum_{r=1}^{N_j} \lambda^j_r 2^{-\frac{jn}{2}} \Phi^j_r, \lambda \in a^s_{p,q}(\mathbb{Z}_\Omega).$$

Moreover, the representation is unique with $\lambda^j_r(f) = 2^{\frac{jn}{2}}(f, \Phi^j_r)$ and

$$\|f|\tilde{A}^s_{p,q}(\Omega)\| \sim \|\lambda|a^s_{p,q}(\mathbb{Z}_\Omega)\|.$$

Wavelet bases in $A^s_{p,q}(\Omega)$

Theorem (T08 - Prop. 3.13)

Let Ω be a Lipschitz (E-thick) domain in $\mathbb{R}^n$. Let $s < 0$ and $u > \sigma_p - s$ resp. $u > \sigma_{p,q} - s$. Then the interior u-wavelet system orthonormal in $L_2(\Omega)$ is a Riesz basis for $B^s_{p,q}(\Omega)$ and $F^s_{p,q}(\Omega)$: For $f \in D'(\Omega)$ it holds

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Proof (main idea) of the theorems: Wavelets serve as atoms and local means. To generate moment conditions for atoms ($s < 0$) or local means ($s > 0$) one has to extend the boundary wavelets. For $s > 0$ hence f needs support in $\bar{\Omega}$.

Wavelet bases in $A^s_{p,q}(\Omega)$, $\tilde{A}^s_{p,q}(\Omega)$ and L_p

Corollary (Homogeneity property - T08, Theorem 2.11.)

Let U_λ be either the balls or cubes with radius resp. diameter λ. Then for $s > \sigma_p$ resp. $s > \sigma_{p,q}$

$$\|f(\lambda\cdot)|\tilde{A}^s_{p,q}(U_1)\| \sim \lambda^{s-\frac{n}{p}}\|f|\tilde{A}^s_{p,q}(U_\lambda)\|.$$

For $s < 0$ it holds

$$\|f(\lambda\cdot)|A^s_{p,q}(U_1)\| \sim \lambda^{s-\frac{n}{p}}\|f|A^s_{p,q}(U_\lambda)\|,$$

where the equivalence constants in both cases are independent of $0 < \lambda \leq 1$ and of f.

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Proof idea: The wavelet representations of $U_{2^{-j}}$ and U_1 can be transformed into each other by dilation with 2^{-j}.

Wavelet bases in the starting stripe

Corollary (T10 - Theorem 2.13., T08 - Prop. 3.21.)

Let Ω be a cellular domain and let $1 \le p < \infty$, $0 < q < \infty$ resp. $1 \le q < \infty$. Then for $u \ge 1$

$$\Phi = \left\{ \Phi_\ell^j : j \in \mathbb{N}_0, \ell = 1, \ldots, N_j \right\} \subset C^u(\Omega)$$

(and the Haar system) are Riesz bases for $A^s_{p,q}(\Omega)$ in the starting stripe

$$\frac{1}{p} - 1 < s < \frac{1}{p}.$$

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Corollary (T10 - Theorem 2.13., T08 - Prop. 3.21.)

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Proof idea: By $A^s_{p,q}(\Omega) = \tilde{A}^s_{p,q}(\Omega)$ the cases with $s \neq 0$ follow from the theorems before. The rest is a matter of interpolation.

Note: For $s > \frac{1}{p}$ there are no **interior** u-wavelet systems for $A^s_{p,q}(\Omega)$ because of boundary values. We have to find "non-interior" wavelet systems.

Traces on the boundary of cubes (i)

Let $Q = \{x \in \mathbb{R}^n : x = (x_1, \dots, x_n), 0 < x_m < 1, m = 1, \dots, n\}$. The boundary $\Gamma = \partial Q$ of Q can be represented as

$$\Gamma = \bigcup_{\ell=0}^{n-1} \Gamma_\ell \text{ with } \Gamma_\ell \cap \Gamma_{\ell'} = \emptyset \text{ for } \ell \neq \ell',$$

where $\Gamma_\ell = \bigcup_{j=0}^{N_\ell} Q_{\ell,j}$ consists of all ℓ-dimensional faces $Q_{\ell,j}$ of Q, which are disjoint cubes of dimension ℓ.

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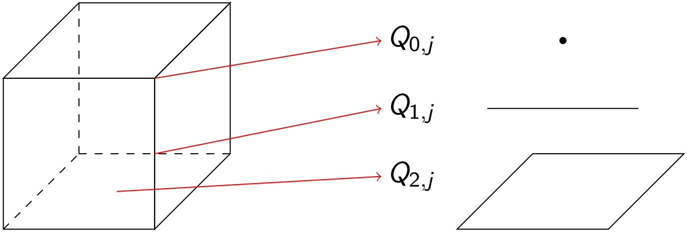

Traces on the boundary of cubes (ii)

Let $tr_{\ell,j}$ be the restriction of $f \in A^s_{p,q}(\mathbb{R}^n)$ to $Q_{\ell,j}$ and

$$tr^r_\ell : f \mapsto TR^r_\ell(f) := \prod \left\{ tr_{\ell,j}\, D^\alpha_\gamma f : |\alpha| \le r,\, j = 0, \ldots, N_\ell \right\},$$

where only derivatives perpendicular to $Q_{\ell,j}$ are admitted.

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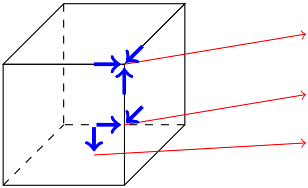
where only derivatives perpendicular to $Q_{\ell,j}$ are admitted. Then we consider the composite mapping for $\bar{r} = \left(r^{\ell_0}, \ldots r^{n-1}\right)$, $\ell_0 \le n-1$ and $r^\ell = \lfloor s - \frac{n-\ell}{p} \rfloor$:

$$tr^{\bar{r}}_\Gamma : f \mapsto \prod_{\ell=\ell_0}^{n-1} TR^{\bar{r}}_\ell(f).$$



$Q_{0,j}$: 3 directions, $1 + \lfloor s - \frac{3}{p} \rfloor$ derivatives

$Q_{1,j}$: 2 directions, $1 + \lfloor s - \frac{2}{p} \rfloor$ derivatives

$Q_{2,j}$: 1 direction, $1 + \lfloor s - \frac{1}{p} \rfloor$ derivatives

Traces on the boundary of cubes (iii)

Theorem

Let

$$1 \leq p < \infty, 0 < q < \infty, s > \frac{1}{p} \text{ and } s - \frac{k}{p} \notin \mathbb{N}_0 \text{ for } k = 1, \ldots, n.$$

Let $\ell_0 = 0$ *if* $s > \frac{n}{p}$. *Otherwise* $\ell_0 \in \mathbb{N}$ *is chosen such that*

$$0 < s - \frac{n - \ell_0}{p} < \frac{1}{p}.$$

Then

$$tr_{\Gamma}^{\bar{r}} : B^s_{p,q}(Q) \mapsto \prod_{\ell=\ell_0}^{n-1} \prod_{j=0}^{N_\ell} \prod_{|\alpha| \leq r^\ell} B_{p,q}^{s - \frac{n-\ell}{p} - |\alpha|}(Q_{\ell,j}),$$

$$tr_{\Gamma}^{\bar{r}} : F^s_{p,q}(Q) \mapsto \prod_{\ell=\ell_0}^{n-1} \prod_{j=0}^{N_\ell} \prod_{|\alpha| \leq r^\ell} B_{p,p}^{s - \frac{n-\ell}{p} - |\alpha|}(Q_{\ell,j}).$$

Traces on the boundary of cubes (iv)

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Then

$$\tilde{B}^s_{p,q}(Q) = \left\{ f \in B^s_{p,q}(Q) : tr^{\bar{r}}_{\Gamma} = 0 \right\},$$
$$\tilde{F}^s_{p,q}(Q) = \left\{ f \in F^s_{p,q}(Q) : tr^{\bar{r}}_{\Gamma} = 0 \right\}.$$

Extension operators for the boundary of cubes (i)

Theorem

Let p, q, s, ℓ_0 as in the theorem before and additionally $s < u$. Then there is a **wavelet-friendly** *extension operator*

$$Ext_\Gamma^{\bar{r},u} : \prod_{\ell=\ell_0}^{n-1} \prod_{j=0}^{N_\ell} \prod_{|\alpha| \le r^\ell} \tilde{B}_{p,q}^{s-\frac{n-\ell}{p}-|\alpha|}(Q_{\ell,j}) \mapsto B_{p,q}^s(Q),$$

$$Ext_\Gamma^{\bar{r},u} : \prod_{\ell=\ell_0}^{n-1} \prod_{j=0}^{N_\ell} \prod_{|\alpha| \le r^\ell} \tilde{B}_{p,p}^{s-\frac{n-\ell}{p}-|\alpha|}(Q_{\ell,j}) \mapsto F_{p,q}^s(Q).$$

It holds

$$tr_\Gamma^{\bar{r}} \circ ext_\Gamma^{\bar{r},u} = id.$$

Extension operators for the boundary of cubes (ii)

Theorem

Furthermore,

$$B^s_{p,q}(Q) = \tilde{B}^s_{p,q}(Q) \times Ext^{\bar{r},u}_{\Gamma} \prod_{\ell=\ell_0}^{n-1} \prod_{j=0}^{N_\ell} \prod_{|\alpha| \le r^\ell} \tilde{B}^{s-\frac{n-\ell}{p}-|\alpha|}_{p,q}(Q_{\ell,j}),$$

$$F^s_{p,q}(Q) = \tilde{F}^s_{p,q}(Q) \times Ext^{\bar{r},u}_{\Gamma} \prod_{\ell=\ell_0}^{n-1} \prod_{j=0}^{N_\ell} \prod_{|\alpha| \le r^\ell} \tilde{B}^{s-\frac{n-\ell}{p}-|\alpha|}_{p,p}(Q_{\ell,j}).$$

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Proof: Use

$$\tilde{B}^s_{p,q}(Q) = \left\{ f \in B^s_{p,q}(Q) : tr^{\bar{r}}_{\Gamma} = 0 \right\},$$

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Wavelets for cubes

Theorem (T08 - Theorem 6.30)

Let $A^s_{p,q}(Q)$ (can be extended to cellular domains) be given with

$$1 \le p < \infty, s > \frac{1}{p} \text{ and } s - \frac{k}{p} \notin \mathbb{N}_0 \text{ for } k = 1, \dots, n, 0 < q < \infty$$

($q \ge 1$ for the F-spaces). Then there is an oscillating u-wavelet system with $u > s$ which is a Riesz basis in $A^s_{p,q}(Q)$.

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($q \geq 1$ for the F-spaces). Then there is an oscillating u-wavelet system with $u > s$ which is a Riesz basis in $A^s_{p,q}(Q)$.

Proof: Use the decomposition

$$B^s_{p,q}(Q) = \tilde{B}^s_{p,q}(Q) \times Ext^{\bar{r},u}_{\Gamma} \prod_{\ell=\ell_0}^{n-1} \prod_{j=0}^{N_\ell} \prod_{|\alpha| \leq r^\ell} \tilde{B}^{s-\frac{n-\ell}{p}-|\alpha|}_{p,q}(Q_{\ell,j})$$

and the fact that every space on the right hand side has a u-wavelet system which is a Riesz basis. Now one must ensure that **wavelet-friendly** extension operators are really wavelet-friendly.

The case $s - \frac{1}{p} \in \mathbb{N}_0$

Let Ω now be a C^∞-domain (so also a cellular domain) and let $1 < p, q < \infty$. By a translation and localization argument one always has

$$\tilde{A}^s_{p,q}(\Omega) \hookrightarrow \mathring{A}^s_{p,q}(\Omega).$$

The converse is true if $s - \frac{1}{p} \notin \mathbb{N}$. Using Hardy inequalities from [T01] one gets

$$\int_\Omega \frac{|f(x)|^p}{d^{sp}(x)}\, dx \leq c \|f|F^s_{p,q}(\mathbb{R}^n)\| \text{ for } f \in \tilde{F}^s_{p,q}(\Omega),$$

where $d(x) = dist(x, \Gamma)$. Then $\chi_\Omega \notin \tilde{F}^s_{p,q}(\Omega)$ for $s \geq \frac{1}{p}$.

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where $d(x) = dist(x, \Gamma)$. Then $\chi_\Omega \notin \tilde{F}^s_{p,q}(\Omega)$ for $s \geq \frac{1}{p}$.

On the other hand $\chi_\Omega \in \mathring{F}^{\frac{1}{p}}_{p,q}(\Omega) = F^{\frac{1}{p}}_{p,q}(\Omega)$ by an atomic decomposition argument.

Analogously, it follows $\tilde{F}^s_{p,q}(\Omega) \neq \mathring{F}^s_{p,q}(\Omega)$ if $s - \frac{1}{p} \in \mathbb{N}$. There are no Riesz frames for these $\mathring{F}^s_{p,q}(\Omega)$ (T08 - Prop. 6.40.).

The cases $s - \frac{k}{p} \in \mathbb{N}_0$ with $k = 2, \ldots, n$

It is known that not all the conditions are necessary depending on the kind of domain.

- For planar C^∞-domains ($n = 2$) the restriction for $k = 2$ is not necessary,
- For C^∞-domains with boundaries diffeomorphic to $\mathbb{S}^n$ the condition for $k = n$ is not necessary,
- For arbitrary C^∞-domains and

$$1 \leq p < \infty, \frac{1}{p} - 1 < s < \frac{2}{p}, s \neq \frac{1}{p}$$

  one finds a Riesz u-wavelet basis without further restrictions since the boundary has a basis,
- For the torus one needs no further conditions since the boundary is equipped with a u-wavelet basis ([T08] - sect. 1.3.2). So $W^1_2(\Omega)$ has a basis if Ω is a torus which is unknown if Ω is a ball and $n \geq 3$.

Reinforced spaces

Let $\Omega_\varepsilon = \{x \in \Omega, d(x) < \varepsilon\}$ and

$$F^{s,rinf}_{p,q}(\Omega) = \begin{cases} F^s_{p,q}(\Omega) \text{ if} & s - \frac{1}{p} \notin \mathbb{N}_0 \\ f \in F^s_{p,q}(\Omega) : d^{-\frac{1}{p}} \frac{\partial^r f}{\partial \nu^r} \in L_p(\Omega_\varepsilon)\} \text{ if} & s - \frac{1}{p} = r \in \mathbb{N}_0. \end{cases}$$

If Ω is a bounded C^∞-domain, then

$$\tilde{F}^s_{p,q}(\Omega) = \left\{ f \in F^{s,rinf}_{p,q}(\Omega) : tr^{r-1}_\Gamma f = 0 \right\}.$$

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If Ω is a bounded C^∞-domain, then

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This and the observation that the known extension operators also map to $F^{s,rinf}_{p,q}(\Omega)$ instead of $F^s_{p,q}(\Omega)$ is the starting point for finding a u-wavelet basis.

Reinforced spaces advanced

Roughly speaking: This gives u-wavelet bases for the modified spaces $F^{s,rinf}_{p,q}(\Omega)$ in the cases of domains where no more restrictions than $s - \frac{1}{p} \notin \mathbb{N}_0$ are necessary.

Reinforced spaces advanced

Roughly speaking: This gives u-wavelet bases for the modified spaces $F^{s,rinf}_{p,q}(\Omega)$ in the cases of domains where no more restrictions than $s - \frac{1}{p} \notin \mathbb{N}_0$ are necessary.

The big question: Can one extend this idea to cellular domains with more restrictions?

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The big question: Can one extend this idea to cellular domains with more restrictions?

The problem: One has to ensure a composition of type

$$\tilde{F}^{s}_{p,q}(\Omega) = \left\{ f \in F^{s,rinf}_{p,q}(\Omega) : tr^{r-1}_{\Gamma} f = 0 \right\}.$$

and has to check if the extension operators are suitable (which they should be). Furthermore, we need conditions for definiting $F^{s,rinf}_{p,q}(\Omega)$ which handle the lower dimensional boundary faces. This has not been done so far. In [T08], Sect. 6.2.4, there is some overview given for $W^1_2(Q)$.

The end

Thank you for your attention

Questions?