

Pointwise multipliers and diffeomorphisms in function spaces

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The problem setting

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The aim:

If φ fulfils . . . , then P_φ resp. D_φ maps the function space A into A .

The spaces C^k

Let C^k be the space of all k -times differentiable functions $f : \mathbb{R}^n \rightarrow \mathbb{R}$ such that

$$\|f\|_{C^k} := \sum_{|\alpha| \leq k} \sup |D^\alpha f(x)| < \infty.$$

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Then

$$f, g \in C^k \Rightarrow f \cdot g \in C^k \text{ and } \|f \cdot g\|_{C^k} \leq c_k \|f\|_{C^k} \cdot \|g\|_{C^k}$$

and

$$(\forall f \in C^k : f \cdot g \in C^k) \Rightarrow g \in C^k \text{ and } \|P_g : C^k \rightarrow C^k\| \geq \|g\|_{C^k}.$$

Proof: Leibniz rule and $1 \in C^k$.

The Hölder spaces \mathcal{C}^k

Let $0 < \sigma \leq 1$ and $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be continuous. We define

$$\|f|lip^\sigma\| := \sup_{x,y \in \mathbb{R}^n, x \neq y} \frac{|f(x) - f(y)|}{|x - y|^\sigma},$$

Let $s > 0$ and $s = \lfloor s \rfloor + \{s\}$ with $\lfloor s \rfloor \in \mathbb{Z}$ and $\{s\} \in (0, 1]$. Then the Hölder space with index s is given by

$$\mathcal{C}^s = \left\{ f \in C^{\lfloor s \rfloor} : \|f|C^s\| := \|f|C^{\lfloor s \rfloor}\| + \sum_{|\alpha|=\lfloor s \rfloor} \|D^\alpha f|lip^{\{s\}}\| < \infty \right\}.$$

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It holds

$$f, g \in \mathcal{C}^s \Rightarrow f \cdot g \in \mathcal{C}^s \text{ and } \|f \cdot g|C^s\| \leq c_s \|f|C^s\| \cdot \|g|C^s\|.$$

and

$$(\forall f \in \mathcal{C}^s : f \cdot g \in \mathcal{C}^s) \Rightarrow g \in \mathcal{C}^s \text{ and } \|P_g : \mathcal{C}^s \rightarrow \mathcal{C}^s\| \geq \|g|C^s\|$$

Proof: Leibniz rule for Hölder spaces and $1 \in \mathcal{C}^s$.

The Lebesgue spaces L_p

Let $0 < p \leq \infty$ and L_p the usual set of equivalence classes of measurable functions f with finite

$$\|f\|_{L_p} := \begin{cases} \left(\int_{\mathbb{R}^n} |f(x)|^p dx \right)^{\frac{1}{p}} & , 0 < p < \infty \\ \text{ess sup } |f(x)| & , p = \infty \end{cases}$$

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Then

$$f \in L_p, g \in L_\infty \Rightarrow f \cdot g \in L_p \text{ and } \|f \cdot g\|_{L_p} \leq \|f\|_{L_p} \cdot \|g\|_{L_\infty}$$

and

$$(\forall f \in L_p : f \cdot g \in L_p) \Rightarrow g \in L_\infty \text{ and } \|P_g : L_p \rightarrow L_p\| \geq \|g\|_{L_\infty}$$

The Sobolev spaces $W_p^k(i)$

Let $1 < p < \infty$, $k \in \mathbb{N}_0$ and W_p^k the set of equivalence classes of measurable functions f with finite

$$\|f|W_p^k\| := \sum_{|\alpha| \leq k} \|D^\alpha f(x)|L_p\|.$$

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Then

$$f \in W_p^k, g \in C^k \Rightarrow f \cdot g \in W_p^k \text{ and } \|f \cdot g|W_p^k\| \leq \|f|W_p^k\| \cdot \|g|C^k\|.$$

The converse is not true!

The Sobolev spaces W_p^k (ii)

Theorem (Sobolev embedding)

Let $k_1 < k_2$ and $k_1 - \frac{n}{p_1} \leq k_2 - \frac{n}{p_2}$. Then

$$W_{p_2}^{k_2} \hookrightarrow W_{p_1}^{k_1}.$$

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Theorem (Multiplier algebra)

If $k > \frac{n}{p}$, then

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Theorem (Multiplier algebra)

If $k > \frac{n}{p}$, then

$$\|f \cdot g\|_{W_p^k} \leq \|f\|_{W_p^k} \cdot \|g\|_{W_p^k}.$$

Proof: We start with

$$\|D^\alpha(f \cdot g)\|_{L_p} \leq c \sum \|(D^\beta f) \cdot (D^{\alpha-\beta} g)\|_{L_p}$$

The Sobolev spaces W_p^k (iii)

$$\begin{aligned}\|D^\alpha(f \cdot g)|_{L_p}\| &\leq c \sum \| (D^\beta f) \cdot (D^{\alpha-\beta} g) \|_{L_p} \\ &\leq c \sum \| (D^\beta f) \|_{L_{p_1}} \cdot \| (D^{\alpha-\beta} g) \|_{L_{p_2}} \\ &\leq c \sum \| f \|_{W_{p_1}^{|\beta|}} \cdot \| g \|_{W_{p_2}^{|\alpha|-|\beta|}} \\ &\leq c' \| f \|_{W_p^k} \cdot \| g \|_{W_p^k}.\end{aligned}$$

The Sobolev spaces W_p^k (iii)

$$\begin{aligned}
 \|D^\alpha(f \cdot g)\|_{L_p} &\leq c \sum \| (D^\beta f) \cdot (D^{\alpha-\beta} g) \|_{L_p} \\
 &\leq c \sum \| (D^\beta f) \|_{L_{p_1}} \cdot \| (D^{\alpha-\beta} g) \|_{L_{p_2}} \\
 &\leq c \sum \| f \|_{W_{p_1}^{|\beta|}} \cdot \| g \|_{W_{p_2}^{|\alpha|-|\beta|}} \\
 &\leq c' \| f \|_{W_p^k} \cdot \| g \|_{W_p^k}.
 \end{aligned}$$

Here ($|\alpha| \leq k$)

$$\begin{aligned}
 \frac{1}{p_1} + \frac{1}{p_2} &= \frac{1}{p} \\
 |\beta| - \frac{n}{p_1} &\leq k - \frac{n}{p} \\
 |\alpha| - |\beta| - \frac{n}{p_2} &\leq k - \frac{n}{p}
 \end{aligned}$$

This is possible, if $k > \frac{n}{p}$.

The Sobolev spaces W_p^k (iv)

Theorem (see e.g Runst and Sickel 1996)

The spaces $W_p^k \cap L_\infty$ are multiplier algebras, even

$$\|f \cdot g\|_{W_p^k} \leq c \left(\|f\|_{W_p^k} \cdot \|g\|_{L_\infty} + \|g\|_{W_p^k} \cdot \|f\|_{L_\infty} \right)$$

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Theorem (see e.g. Triebel 2008)

If W_p^k is a multiplier algebra, then φ is a pointwise multiplier for W_p^k iff

$$\sup_{m \in \mathbb{Z}} \|\psi(\cdot - m) \cdot \varphi|_{W_p^k}\| < \infty,$$

where ψ is a nonnegative C_0^∞ -function with

$$\sum_m \psi(x - m) = 1 \text{ for } x \in \mathbb{R}^n.$$

Resolution of unity

Let $\varphi_0 \in \mathcal{S}(\mathbb{R}^n)$ such that $\text{supp } \varphi_0 \subset \{|x| \leq \frac{3}{2}\}$ and $\varphi_0(x) = 1$ for $|x| \leq 1$. We define

$$\varphi(x) := \varphi_0(x) - \varphi_0(2x) \text{ and } \varphi_j(x) := \varphi(2^{-j}x) \text{ for } j \in \mathbb{N}.$$

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Then we have

$$\begin{aligned} \sum_{j=0}^{\infty} \varphi_j(x) &= 1. \\ |D^\alpha \varphi_j(x)| &\leq c_\alpha 2^{-j|\alpha|}, \\ \text{supp } \varphi_j &\subset \{2^{j-1} \leq |x| \leq 2^{j+1}\}, \end{aligned} \tag{1}$$

A sequence of functions $\{\varphi_j\}_{j=0}^{\infty}$ with (1), $\varphi_j \in \mathcal{S}(\mathbb{R}^n)$ and φ_0 as above will be called resolution of unity.

The definition of $B_{p,q}^s(\mathbb{R}^n)$

Let $\{\varphi_j\}_{j=0}^\infty$ be a resolution of unity. Let $0 < p \leq \infty$, $0 < q \leq \infty$ and $s \in \mathbb{R}$. For $f \in \mathcal{S}'(\mathbb{R}^n)$ we define

$$\|f|B_{p,q}^s(\mathbb{R}^n)\|^\varphi := \left(\sum_{j=0}^{\infty} 2^{jsq} \|(\varphi_j \hat{f})^\vee\|_{L_p}^q \right)^{\frac{1}{q}}$$

(modified in case $q = \infty$) and

$$B_{p,q}^{s,\varphi}(\mathbb{R}^n) := \{f \in \mathcal{S}'(\mathbb{R}^n) : \|f|B_{p,q}^s(\mathbb{R}^n)\|^\varphi < \infty\}.$$

Then $(B_{p,q}^{s,\varphi}(\mathbb{R}^n), \|\cdot|B_{p,q}^s(\mathbb{R}^n)\|^\varphi)$ is a quasi-Banach space. It does not depend on the choice of the resolution of unity $\{\varphi_j\}_{j=0}^\infty$ in the sense of equivalent norms. So we denote it shortly by $B_{p,q}^s(\mathbb{R}^n)$.

The definition of $F_{p,q}^s(\mathbb{R}^n)$

Let $\{\varphi_j\}_{j=0}^\infty$ be a resolution of unity. Let $0 < p < \infty$, $0 < q \leq \infty$ and $s \in \mathbb{R}$. For $f \in \mathcal{S}'(\mathbb{R}^n)$ we define

$$\|f\|_{F_{p,q}^s(\mathbb{R}^n)}^\varphi := \left\| \left(\sum_{j=0}^{\infty} 2^{jsq} |(\varphi_j \hat{f})^\vee|^q \right)^{\frac{1}{q}} \right\|_{L_p}$$

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Atomic characterization of $B_{p,q}^s(\mathbb{R}^n)$

Theorem

Let $0 < p \leq \infty$, $0 < q \leq \infty$ and $s \in \mathbb{R}$. Let $K, L \geq 0$, $K > s$ and $L > \sigma_p - s$. Then $f \in \mathcal{S}'(\mathbb{R}^n)$ belongs to $B_{p,q}^s(\mathbb{R}^n)$ if and only if it can be represented as

$$f = \sum_{\nu=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{\nu,m} \cdot a_{\nu,m} \quad \text{with convergence in } \mathcal{S}'(\mathbb{R}^n).$$

Here $a_{\nu,m}$ are $(s, p)_{K,L}$ -atoms located at $Q_{\nu,m}$ and $\|\lambda|b_{p,q}\| < \infty$. Furthermore, we have in the sense of equivalence of norms

$$\|f|B_{p,q}^s(\mathbb{R}^n)\| \sim \inf \|\lambda|b_{p,q}\|,$$

where the infimum on the right-hand side is taken over all admissible representations of f .

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Treatment of products using atomic decompositions

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$$\varphi \cdot f = \sum_{\nu=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{\nu,m} \cdot \varphi \cdot a_{\nu,m}$$

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If $\varphi \cdot a_{\nu,m}$ are atoms: $\varphi \cdot f \in A_{p,q}^s(\mathbb{R}^n)$

The definition of atoms

A function $a : \mathbb{R}^n \rightarrow \mathbb{R}$ is called classical $(s, p)_{K,L}$ -atom located at $Q_{\nu,m}$ if

$$\text{supp } a \subset d \cdot Q_{\nu,m}$$

$$|D^\alpha a(x)| \leq C \cdot 2^{-\nu\left(s - \frac{n}{p}\right) + |\alpha|\nu} \text{ for all } |\alpha| < K + 1, \quad (2)$$

$$\int_{\mathbb{R}^n} x^\beta a(x) dx = 0 \text{ for all } |\beta| < L. \quad (3)$$

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A function $a : \mathbb{R}^n \rightarrow \mathbb{R}$ is called $(s, p)_{K,L}$ -atom located at $Q_{\nu,m}$ if instead of (2) and (3) it holds (for all $\psi \in \mathcal{C}^L$)

$$\|a(2^{-\nu} \cdot) | \mathcal{C}^K \| \leq C \cdot 2^{-\nu(s - \frac{n}{p})}$$

$$\left| \int_{d \cdot Q_{\nu,m}} \psi(x) a(x) dx \right| \leq C \cdot 2^{-\nu(s + L + n(1 - \frac{1}{p}))} \|\psi\|_{\mathcal{C}^L}$$

Atomic representations revisited

Every classical $(s, p)_{K,L}$ -atom is an $(s, p)_{K,L}$ -atom.

Theorem

The atomic representation theorem for $B_{p,q}^s(\mathbb{R}^n)$ and $F_{p,q}^s(\mathbb{R}^n)$ is valid with both forms of atoms. Hence every f which can be represented as a linear combination of classical $(s, p)_{K,L}$ -atom resp. $(s, p)_{K,L}$ -atom belongs to $B_{p,q}^s(\mathbb{R}^n)$ resp. $F_{p,q}^s(\mathbb{R}^n)$. Hereby

$$K > s \quad \text{and}$$

$$L > \sigma_p - s = \sigma_p = n \left(\frac{1}{p} - 1 \right)_+ - s \quad \text{resp.}$$

$$L > \sigma_{p,q} - s = n \left(\frac{1}{\min(p, q)} - 1 \right)_+ - s$$

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The proof for classical atoms goes back to Triebel '97. The modifications were suggested by Skrzypczak '98, Triebel/Winkelvoss '96.

The pointwise multiplier theorem (i)

Now we get

Lemma

There exists a constant c with the following property: For all $\nu \in \mathbb{N}_0$, $m \in \mathbb{Z}$, all $(s, p)_{K,L}$ -atoms $a_{\nu,m}$ with support in $d \cdot Q_{\nu,m}$ and all $\varphi \in C^p$ with $\rho \geq \max(K, L)$ the product

$$c \cdot \|\varphi\|_{C^p}^{-1} \cdot \varphi \cdot a_{\nu,m}$$

is an $(s, p)_{K,L}$ -atom with support in $d \cdot Q_{\nu,m}$.

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is an $(s, p)_{K,L}$ -atom with support in $d \cdot Q_{\nu,m}$.

Proof: Use that C^ρ is a multiplication algebra.

This does not work for classical atoms $(s, p)_{K,L}$ -atoms with $L \geq 1$, since in general moment conditions are destroyed when multiplying by φ !

The pointwise multiplier theorem (ii)

We get as a Corollary

Theorem

Let $s \in \mathbb{R}$ and $0 < q \leq \infty$.

(i) Let $0 < p \leq \infty$ and $\rho > \max(s, \sigma_p - s)$. Then there exists a positive number c such that

$$\|\varphi f|B_{p,q}^s(\mathbb{R}^n)\| \leq c \|\varphi|C^{\rho}\| \cdot \|f|B_{p,q}^s(\mathbb{R}^n)\|$$

for all $\varphi \in C^{\rho}$ and all $f \in B_{p,q}^s(\mathbb{R}^n)$.

(ii) Let $0 < p < \infty$ and $\rho > \max(s, \sigma_{p,q} - s)$. Then there exists a positive number c such that

$$\|\varphi f|F_{p,q}^s(\mathbb{R}^n)\| \leq c \|\varphi|C^{\rho}\| \cdot \|f|F_{p,q}^s(\mathbb{R}^n)\|$$

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The diffeomorphism theorem (i)

In the same way we can treat the mapping D_φ :

$$f = \sum_{\nu=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{\nu,m} \cdot a_{\nu,m} \Rightarrow f \circ \varphi = \sum_{\nu=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{\nu,m} \cdot (a_{\nu,m} \circ \varphi).$$

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Hence we have to investigate if $a_{\nu,m} \circ \varphi$ is an $(s, p)_{K,L}$ -atom when $a_{\nu,m}$ is an $(s, p)_{K,L}$ -atom.

The diffeomorphism theorem (ii)

Definition

Let $\rho \geq 1$.

(i) Let $\rho = 1$. We say that the map $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a 1-diffeomorphism if φ is a bi-Lipschitzian map, i.e. that there are constants $c_1, c_2 > 0$ such that

$$c_1 \leq \frac{|\varphi(x) - \varphi(y)|}{|x - y|} \leq c_2.$$

for all $x, y \in \mathbb{R}^n$ with $0 < |x - y| \leq 1$.

(ii) Let $\rho > 1$. We say that the one-to-one map $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a ρ -diffeomorphism if the n components φ_i of φ have classical derivatives up to order $[\rho]$ with $\frac{\partial \varphi_i}{\partial x_j} \in \mathcal{C}^{\rho-1}$ and $|\det J(\varphi)(x)| \geq c > 0$ for all $x \in \mathbb{R}^n$.

The diffeomorphism theorem (iii)

Theorem

(i) Let $0 < p \leq \infty$, $\rho \geq 1$ and $\rho > \max(s, 1 + \sigma_p - s)$. If φ is a ρ -diffeomorphism, then there exists a constant c such that

$$\|f(\varphi(\cdot))\|_{B_{p,q}^s(\mathbb{R}^n)} \leq c \|f\|_{B_{p,q}^s(\mathbb{R}^n)}$$

for all $f \in B_{p,q}^s(\mathbb{R}^n)$. Hence D_φ maps $B_{p,q}^s(\mathbb{R}^n)$ onto $B_{p,q}^s(\mathbb{R}^n)$.

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(ii) Let $0 < p < \infty$, $\rho \geq 1$ and $\rho > \max(s, 1 + \sigma_{p,q} - s)$. If φ is a ρ -diffeomorphism, then there exists a constant c such that

$$\|f(\varphi(\cdot))\|_{F_{p,q}^s(\mathbb{R}^n)} \leq c \|f\|_{F_{p,q}^s(\mathbb{R}^n)}.$$

for all $f \in F_{p,q}^s(\mathbb{R}^n)$. Hence D_φ maps $F_{p,q}^s(\mathbb{R}^n)$ onto $F_{p,q}^s(\mathbb{R}^n)$.

Proof: Show that $a_{\nu,m}$ is an $(s, p)_{K,L}$ -atom, control the support of the atoms and verify convergence

The end

Thank you for your attention

Questions?