Wavelet bases for function spaces on cellular domains

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(i) Let $2 \leq n$. A special Lipschitz ($C^\infty$-) domain in $\mathbb{R}^n$ is the collection of all points $x = (x', x_n)$ with $x' \in \mathbb{R}^{n-1}$ and $h(x') < x_n$, where $h$ is a Lipschitz (bounded $C^\infty$-) function.
(i) Let $2 \leq n$. A special Lipschitz ($C^{\infty}$-) domain in $\mathbb{R}^n$ is the collection of all points $x = (x', x_n)$ with $x' \in \mathbb{R}^{n-1}$ and $h(x') < x_n$, where $h$ is a Lipschitz (bounded $C^{\infty}$-) function.

(ii) Let $2 \leq n$. A bounded Lipschitz ($C^{\infty}$-) domain is a bounded domain $\Omega$ where the boundary $\Gamma = \partial \Omega$ can be covered by finitely many open balls $B_j$ centred at $\Gamma$ such that $B_j \cap \Omega = B_j \cap \Omega_j$, where $\Omega_j$ are rotations of suitable special Lipschitz ($C^{\infty}$-) domains.
Introduction and basic definitions

Wavelets for function spaces on domains $\Omega$

The exceptional values of $s$

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**Domains in $\mathbb{R}^n$**

(i) Let $2 \leq n$. A special Lipschitz ($C^\infty$-) domain in $\mathbb{R}^n$ is the collection of all points $x = (x', x_n)$ with $x' \in \mathbb{R}^{n-1}$ and $h(x') < x_n$, where $h$ is a Lipschitz (bounded $C^\infty$-) function.

(ii) Let $2 \leq n$. A bounded Lipschitz ($C^\infty$-) domain is a bounded domain $\Omega$ where the boundary $\Gamma = \partial \Omega$ can be covered by finitely many open balls $B_j$ centred at $\Gamma$ such that $B_j \cap \Omega = B_j \cap \Omega_j$, where $\Omega_j$ are rotations of suitable special Lipschitz ($C^\infty$-) domains.

(iii) Let $2 \leq n$. A domain $\Omega$ in $\mathbb{R}^n$ is called cellular if it is a bounded Lipschitz domain which can be represented as

$$
\Omega = \left( \bigcup_{\ell=1}^{L} \bar{\Omega}_\ell \right)^{\circ} \text{ with } \Omega_\ell \cap \Omega_{\ell'} = \emptyset \text{ if } \ell \neq \ell'
$$

such that each $\Omega_\ell$ is diffeomorphic to a polyhedron (cube).
The definition of $B^s_{p,q}(\mathbb{R}^n)$

Let $\{\varphi_j\}_{j=0}^{\infty}$ be a resolution of unity. Let $0 < p \leq \infty$, $0 < q \leq \infty$ and $s \in \mathbb{R}$. For $f \in S'(\mathbb{R}^n)$ we define

$$
\| f | B^s_{p,q}(\mathbb{R}^n) \|_\varphi := \left( \sum_{j=0}^{\infty} 2^{jsq} \| (\varphi_j \hat{f}) \|_{L^p}^q \right)^{\frac{1}{q}}
$$

(modified in case $q = \infty$) and

$$
B^s_{p,q}(\mathbb{R}^n) := \{ f \in S'(\mathbb{R}^n) : \| f | B^s_{p,q}(\mathbb{R}^n) \|_\varphi < \infty \}.
$$

Then $(B^s_{p,q}(\mathbb{R}^n), \| \cdot | B^s_{p,q}(\mathbb{R}^n) \|_\varphi)$ is a quasi-Banach space. It does not depend on the choice of the resolution of unity $\{\varphi_j\}_{j=0}^{\infty}$ in the sense of equivalent norms. So we denote it shortly by $B^s_{p,q}(\mathbb{R}^n)$. 
The definition of $F^s_{p,q}(\mathbb{R}^n)$

Let $\{\varphi_j\}_{j=0}^\infty$ be a resolution of unity. Let $0 < p < \infty$, $0 < q \leq \infty$ and $s \in \mathbb{R}$. For $f \in \mathcal{S}'(\mathbb{R}^n)$ we define

$$
\| f | F^s_{p,q}(\mathbb{R}^n) | \varphi := \left\| \left( \sum_{j=0}^\infty 2^{jsq} \| (\varphi_j \hat{f})_E \|_q \right)^{\frac{1}{q}} \|_{L^p} \right\|
$$

(modified in case $q = \infty$) and

$$
F^s_{p,q}(\mathbb{R}^n) := \left\{ f \in \mathcal{S}'(\mathbb{R}^n) : \| f | F^s_{p,q}(\mathbb{R}^n) | \varphi < \infty \right\}.
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Then $(F^s_{p,q}(\mathbb{R}^n), \| \cdot | F^s_{p,q}(\mathbb{R}^n) | \varphi)$ is a quasi-Banach space. It does not depend on the choice of the resolution of unity $\{\varphi_j\}_{j=0}^\infty$ in the sense of equivalent norms. So we denote it shortly by $F^s_{p,q}(\mathbb{R}^n)$. 
The definition of $A^s_{p,q}(\Omega), \tilde{A}^s_{p,q}(\Omega)$ and $\hat{A}^s_{p,q}(\Omega)$ (i)

Let $\Omega$ be an open set in $\mathbb{R}^n$. Denote by $g|\Omega \in D'(\Omega)$ its restriction to $\Omega$, hence $(g|\Omega)(\varphi) = g(\varphi)$ for $\varphi \in D(\Omega)$.

$$A^s_{p,q}(\Omega) := \{ f \in D'(\Omega) : f = g|\Omega \text{ for some } g \in A^s_{p,q}(\mathbb{R}^n) \},$$

$$\|f|A^s_{p,q}(\Omega)\| = \inf \|g|A^s_{p,q}(\mathbb{R}^n)\|,$$

where the infimum is taken over all $g \in A^s_{p,q}(\mathbb{R}^n)$ with $g|\Omega = f$. 

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The definition of $A_{p,q}^s(\Omega)$, $\tilde{A}_{p,q}^s(\Omega)$ and $\hat{A}_{p,q}^s(\Omega)$ (i)

Let $\Omega$ be an open set in $\mathbb{R}^n$. Denote by $g|\Omega \in D'(\Omega)$ its restriction to $\Omega$, hence $(g|\Omega)(\varphi) = g(\varphi)$ for $\varphi \in D(\Omega)$.

\[ A_{p,q}^s(\Omega) := \{ f \in D'(\Omega) : f = g|\Omega \text{ for some } g \in A_{p,q}^s(\mathbb{R}^n) \}, \]

\[ \| f|A_{p,q}^s(\Omega) \| = \inf \| g|A_{p,q}^s(\mathbb{R}^n) \|, \]

where the infimum is taken over all $g \in A_{p,q}^s(\mathbb{R}^n)$ with $g|\Omega = f$.

Moreover, let

\[ \tilde{A}_{p,q}^s(\Omega) := \{ f \in A_{p,q}^s(\mathbb{R}^n) : \text{supp } f \in \Omega \} \]

with the quasi-norm from $A_{p,q}^s(\mathbb{R}^n)$. Then

\[ \tilde{A}_{p,q}^s(\Omega) := \{ f \in D'(\Omega) : f = g|\Omega \text{ for some } g \in \tilde{A}_{p,q}^s(\Omega) \}, \]

\[ \| f|\tilde{A}_{p,q}^s(\Omega) \| = \inf \| g|\tilde{A}_{p,q}^s(\Omega) \| \]

where the infimum is taken over all $g \in \tilde{A}_{p,q}^s(\Omega)$ with $g|\Omega = f$. 

The definition of $A^s_{p,q}(\Omega)$, $\tilde{A}^s_{p,q}(\Omega)$ and $\hat{A}^s_{p,q}(\Omega)$ (ii)

It holds

$$A^s_{p,q}(\Omega) = A^s_{p,q}(\mathbb{R}^n) / \tilde{A}^s_{p,q}(\Omega^c),$$

$$\tilde{A}^s_{p,q}(\Omega) = \tilde{A}^s_{p,q}(\bar{\Omega}) / \hat{A}^s_{p,q}(\partial\Omega)$$

in the sense of quotient spaces and norms. Hence $A^s_{p,q}(\Omega)$ and $\tilde{A}^s_{p,q}(\Omega)$ are quasi-Banach spaces.
The definition of $A_{p,q}^s(\Omega)$, $\tilde{A}_{p,q}^s(\Omega)$ and $\check{A}_{p,q}^s(\Omega)$ (ii)

It holds

\[
A_{p,q}^s(\Omega) = A_{p,q}^s(\mathbb{R}^n) / \tilde{A}_{p,q}^s(\Omega^C),
\]

\[
\tilde{A}_{p,q}^s(\Omega) = \tilde{A}_{p,q}^s(\Omega) / \tilde{A}_{p,q}^s(\partial\Omega)
\]

in the sense of quotient spaces and norms. Hence $A_{p,q}^s(\Omega)$ and $\tilde{A}_{p,q}^s(\Omega)$ are quasi-Banach spaces.

Let $\Omega$ be a cellular domain and let

\[
0 < p \leq \infty, 0 < q \leq \infty, \max \left( n \left( \frac{1}{p} - 1 \right), \frac{1}{p} - 1 \right) < s
\]

($p < \infty$ for the $F$-spaces). Then by T08, Prop. 6.13.

\[
\tilde{A}_{p,q}^s(\Omega) = \tilde{A}_{p,q}^s(\overline{\Omega}) \text{ in the sense of } \tilde{A}_{p,q}^s(\partial\Omega) = \{0\}.
\]
The definition of $A^s_{p,q}(\Omega)$, $\tilde{A}^s_{p,q}(\Omega)$ and $\hat{A}^s_{p,q}(\Omega)$ (iii)

Furthermore, let $\hat{A}^s_{p,q}(\Omega)$ be the completion of $D(\Omega)$ with respect to $\| \cdot \|_{A^s_{p,q}(\Omega)}$.

Theorem (The starting stripe - T08, Prop. 6.13.)

Let $\Omega$ be a cellular domain and let

$$0 < p < \infty, 0 < q < \infty, \max \left( n \left( \frac{1}{p} - 1 \right), \frac{1}{p} - 1 \right) < s < \frac{1}{p}$$

Then

$$A^s_{p,q}(\Omega) = \hat{A}^s_{p,q}(\Omega) = \tilde{A}^s_{p,q}(\Omega).$$
Furthermore, let $\mathring{A}^s_{p,q}(\Omega)$ be the completion of $D(\Omega)$ with respect to $\| \cdot |A^s_{p,q}(\Omega)\|$. 

**Theorem (The starting stripe - T08, Prop. 6.13.)**

Let $\Omega$ be a cellular domain and let

$$0 < p < \infty, 0 < q < \infty, \max \left( n \left( \frac{1}{p} - 1 \right), \frac{1}{p} - 1 \right) < s < \frac{1}{p}$$

Then

$$A^s_{p,q}(\Omega) = \mathring{A}^s_{p,q}(\Omega) = \mathring{\mathring{A}}^s_{p,q}(\Omega).$$

Proof: Use that $\chi_\Omega$ is a pointwise multiplier in these spaces.
The starting stripe

![Graph showing the starting stripe with the equation \( s = \frac{n+1}{n} \) for certain values of \( n \).]
Wavelets for function spaces on domains $\Omega$ (i)

From now on let $\Omega$ be a cellular domain and $\Gamma = \partial \Omega$. Let

$$Z^\Omega = \left\{ x^j_\ell \in \Omega : j \in \mathbb{N}_0, \ell = 1, \ldots, N_j \right\},$$

(typically $N_j \sim 2^{jn}$), such that for some $c_1 > 0$

$$|x^j_\ell - x^{j}_{\ell'}| \geq c_1 2^{-j}, j \in \mathbb{N}_0, \ell \neq \ell'.$$
Wavelets for function spaces on domains $\Omega$ (i)

From now on let $\Omega$ be a cellular domain and $\Gamma = \partial \Omega$. Let

$$\mathbb{Z}^\Omega = \left\{ x^j_\ell \in \Omega : j \in \mathbb{N}_0, \ell = 1, \ldots, N_j \right\},$$

(typically $N_j \sim 2^{jn}$), such that for some $c_1 > 0$

$$|x^j_\ell - x^j_{\ell'}| \geq c_1 2^{-j}, \quad j \in \mathbb{N}_0, \quad \ell \neq \ell'.$$

We can introduce the usual sequence spaces $b^s_{p,q}(\mathbb{Z}^\Omega)$ and $f^s_{p,q}(\mathbb{Z}^\Omega)$ adapted to $\mathbb{Z}^\Omega$ and abbreviated by $a^s_{p,q}(\mathbb{Z}^\Omega)$. For example,

$$\| \lambda | f^s_{p,q}(\mathbb{Z}^\Omega) \| := \left\| \left( \sum_{j=0}^{\infty} \sum_{\ell=1}^{N_j} 2^{jsq} |\chi^j_\ell \chi_{j,\ell}|^q \right)^{\frac{1}{q}} \right\|_{L_p(\Omega)}^{\frac{1}{q}},$$

where $\chi_{j,\ell}$ is the characteristic function of $B(x^j_\ell, c_3 2^{-j})$ or $B(x^j_\ell, c_3 2^{-j}) \cap \bar{\Omega}$ with another constant $c_3$. 

Wavelets for function spaces on domains $\Omega$ (ii)

Let $u \in \mathbb{N}_0$ (now incorporating Haar-Wavelets). Then

$$\Phi = \left\{ \Phi^j_\ell : j \in \mathbb{N}_0, \ell = 1, \ldots, N_j \right\} \in C^u(\Omega)$$

is called a u-wavelet system in $\bar{\Omega}$ if it fulfils

- support conditions: For some $c_3 > 0$ let

$$\text{supp } \Phi^j_\ell \subset B(x^j_\ell, c_3 2^{-j}) \cap \bar{\Omega}, j \in \mathbb{N}_0, \ell = 1, \ldots, N_j,$$
Let $u \in \mathbb{N}_0$ (now incorporating Haar-Wavelets). Then

$$\Phi = \left\{ \Phi^j_\ell : j \in \mathbb{N}_0, \ell = 1, \ldots, N_j \right\} \in C^u(\Omega)$$

is called a $u$-wavelet system in $\bar{\Omega}$ if it fulfils

- support conditions: For some $c_3 > 0$ let

  $$\text{supp} \ \Phi^j_\ell \subset B(x^j_\ell, c_3 2^{-j}) \cap \bar{\Omega}, \ j \in \mathbb{N}_0, \ell = 1, \ldots, N_j,$$

- derivative conditions: For some $c_4 > 0$ and all $\alpha \in \mathbb{N}^n_0$ with $0 \leq |\alpha| \leq u$ let

  $$\left| D^\alpha \Phi^j_\ell(x) \right| \leq c_4 2^{j + |\alpha|}, \ j \in \mathbb{N}_0, \ell = 1, \ldots, N_j, x \in \Omega.$$
Additionally, the u-wavelet system is called oscillating if it fulfils

(substitute) moment conditions: Let $c_5$ and $c_6 < c_7$ be constants such that

$$\text{dist}(B(x_0^\ell, c_3), \Gamma) \geq c_6, \text{ for } \ell = 1, \ldots, \mathbb{N}_0 \text{ and}$$

$$\left| \int_{\Omega} \psi(x) \Phi^j_\ell(x) \, dx \right| \leq c_5 2^{-j/2} c_6^{-j} \| \psi \| C^u(\Omega) \| \text{ for all } \psi \in C^u(\Omega)$$

for all $\Phi^j_\ell$ with $j \in \mathbb{N}$ and $c_6 2^{-j} \leq \text{dist}(B(x_0^\ell, c_3), \Gamma) \leq c_7 2^{-j}$. 
Additionally, the u-wavelet system is called oscillating if it fulfils

- (substitute) moment conditions: Let $c_5$ and $c_6 < c_7$ be constants such that

$$\text{dist}(B(x_0^\ell, c_3), \Gamma) \geq c_6, \text{ for } \ell = 1, \ldots, N_0 \text{ and}$$

$$\left| \int_{\Omega} \psi(x) \Phi_j^\ell(x) \, dx \right| \leq c_5 2^{-j \frac{n}{2} - ju} \|\psi\| C^u(\Omega) \| \text{ for all } \psi \in C^u(\Omega)$$

for all $\Phi_j^\ell$ with $j \in \mathbb{N}$ and $c_6 2^{-j} \leq \text{dist}(B(x_j^0, c_3), \Gamma) \leq c_7 2^{-j}$.

An oscillating u-wavelet system is called interior if it fulfils

- (further) interior support conditions, namely

$$\text{dist}(B(x_j^\ell, c_3 2^{-j}), \Gamma) \geq c_6 2^{-j}, j \in \mathbb{N}_0, \ell = 1, \ldots, N_j.$$
Wavelet bases in $L_2(\Omega)$ - the starting point

**Theorem (T08 - Theorem 2.33)**

Let $\Omega$ be an arbitrary domain in $\mathbb{R}^n$. For any $u \in \mathbb{N}_0$ there is an orthonormal basis in $L_2(\Omega)$ which is simultaneously an interior $u$-wavelet system. It even fulfils “true” moment conditions inside the domain.

For $u = 0$ one can take the Haar Wavelet suitably restricted.
Theorem (T08 - Theorem 2.33)

Let $\Omega$ be an arbitrary domain in $\mathbb{R}^n$. For any $u \in \mathbb{N}_0$ there is an orthonormal basis in $L_2(\Omega)$ which is simultaneously an interior $u$-wavelet system. It even fulfills “true” moment conditions inside the domain.

For $u = 0$ one can take the Haar Wavelet suitably restricted.

Proof (main idea): First start with wavelets with small enough support. Take a wavelet basis in $L_2(\mathbb{R}^n)$, restrict them suitably to $\Omega$ adapted to a Whitney decomposition. Orthonormalize at the boundary (losing moment conditions there).

In the special case of the Haar Wavelet no further orthonormalisation is necessary because of no overlapping.
Wavelet bases in $\tilde{A}^{s}_{p,q}(\Omega)$ and $L_p$

**Theorem (T08 - Theorem 2.36, Prop. 3.10)**

Let $\Omega$ be a Lipschitz (E-thick) domain in $\mathbb{R}^n$. Let $u > s > \sigma_p$ resp. $> \sigma_{p,q}$. Then the interior $u$-wavelet system orthonormal in $L_2(\Omega)$ is a Riesz basis for $L_p(\Omega) = \tilde{F}^0_{p,2}(\Omega), 1 < p < \infty$, for $\tilde{B}^s_{p,q}(\Omega)$ and $\tilde{F}^s_{p,q}(\Omega)$. This means $f \in L^1_{\text{loc}}$ (resp. $D'(\Omega)$) is an element of $\tilde{A}^s_{p,q}(\Omega)$ if and only if it can be represented as

$$f = \sum_{j=0}^{\infty} \sum_{r=1}^{N_j} \lambda_r^j 2^{-jn/2} \Phi_r^j, \lambda \in a^s_{p,q}(\mathbb{Z}_\Omega).$$

Moreover, the representation is unique with $\lambda_r^j(f) = 2^{jn/2} (f, \Phi_r^j)$ and

$$\| f | \tilde{A}^s_{p,q}(\Omega) \| \sim \| \lambda | a^s_{p,q}(\mathbb{Z}_\Omega) \|.$$
Wavelet bases in $A_{p,q}^s(\Omega)$

**Theorem (T08 - Prop. 3.13)**

Let $\Omega$ be a Lipschitz (E-thick) domain in $\mathbb{R}^n$. Let $s < 0$ and $u > \sigma_p - s$ resp. $u > \sigma_{p,q} - s$. Then the interior $u$-wavelet system orthonormal in $L_2(\Omega)$ is a Riesz basis for $B_{p,q}^s(\Omega)$ and $F_{p,q}^s(\Omega)$. This means $f \in D'(\Omega))$ is an element of $A_{p,q}^s(\Omega)$ if and only if it can be represented as

$$f = \sum_{j=0}^{\infty} \sum_{r=1}^{N_j} \lambda^j_r 2^{-jn/2} \Phi^j_r, \lambda \in a_{p,q}^s(\mathbb{Z}_\Omega).$$

Moreover, the representation is unique with $\lambda^j_r(f) = 2^{jn/2} (f, \Phi^j_r)$ and

$$\|f| A_{p,q}^s(\Omega)\| \sim \|\lambda| a_{p,q}^s(\mathbb{Z}_\Omega)\|.$$
Wavelet bases in $A^s_{p,q}(\Omega)$, $\tilde{A}^s_{p,q}(\Omega)$ and $L_p$

Proof (main idea) of the theorems: Wavelets serve as atoms and local means. To generate moment conditions for atoms ($s < 0$) or local means ($s > 0$) one has to extend the boundary wavelets. Then $f$ needs support in $\bar{\Omega}$ for $s > 0$. 
Proof (main idea) of the theorems: Wavelets serve as atoms and local means. To generate moment conditions for atoms ($s < 0$) or local means ($s > 0$) one has to extend the boundary wavelets. Then $f$ needs support in $\bar{\Omega}$ for $s > 0$.

Corollary (Homogeneity property - T08, Theorem 2.11.)

Let $U_\lambda$ be either the balls or cubes with radius resp. diameter $\lambda$. Then for $s > \sigma_p$ resp. $s > \sigma_{p,q}$

$$\| f(\lambda \cdot) | \tilde{A}^s_{p,q}(U_1) \| \sim \lambda^{s - \frac{n}{p}} \| f | \tilde{A}^s_{p,q}(U_\lambda) \|.$$  

For $s < 0$ it holds

$$\| f(\lambda \cdot) | A^s_{p,q}(U_1) \| \sim \lambda^{s - \frac{n}{p}} \| f | A^s_{p,q}(U_\lambda) \|,$$

where the equivalence constants in both cases are independent of $0 < \lambda \leq 1$ and of $f$. 
Wavelet bases in $A^s_{p,q}(\Omega)$, $\tilde{A}^s_{p,q}(\Omega)$ and $L_p$

Proof (main idea) of the theorems: Wavelets serve as atoms and local means. To generate moment conditions for atoms ($s < 0$) or local means ($s > 0$) one has to extend the boundary wavelets. Then $f$ needs support in $\bar{\Omega}$ for $s > 0$.

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Let $U_\lambda$ be either the balls or cubes with radius resp. diameter $\lambda$. Then for $s > \sigma_p$ resp. $s > \sigma_{p,q}$

$$\| f(\cdot) |\tilde{A}^s_{p,q}(U_1) \| \sim \lambda^{s-n/p} \| f|\tilde{A}^s_{p,q}(U_\lambda) \| .$$

For $s < 0$ it holds

$$\| f(\cdot) |A^s_{p,q}(U_1) \| \sim \lambda^{s-n/p} \| f|A^s_{p,q}(U_\lambda) \| ,$$

where the equivalence constants in both cases are independent of $0 < \lambda \leq 1$ and of $f$.

Proof idea: The wavelet representations of $U_{2^{-j}}$ and $U_1$ can be transformed into each other by dilation with $2^{-j}$. 
Wavelet bases in the starting stripe

Corollary (T09 - Theorem 2.13., T08 - Prop. 3.21.)

Let $\Omega$ be a cellular domain and let $1 \leq p < \infty$, $0 < q < \infty$ resp. $1 \leq q < \infty$. Then the interior $u$-wavelet system (and the Haar Wavelet system) are Riesz bases for $A_{p,q}^s(\Omega)$ in the starting stripe

$$\frac{1}{p} - 1 < s < \frac{1}{p}.$$
Wavelet bases in the starting stripe

Corollary (T09 - Theorem 2.13., T08 - Prop. 3.21.)

Let $\Omega$ be a cellular domain and let $1 \leq p < \infty$, $0 < q < \infty$ resp. $1 \leq q < \infty$. Then the interior $u$-wavelet system (and the Haar Wavelet system) are Riesz bases for $A^{s}_{p,q}(\Omega)$ in the starting stripe

\[ \frac{1}{p} - 1 < s < \frac{1}{p}. \]

Proof idea: By $A^{s}_{p,q}(\Omega) = \tilde{A}^{s}_{p,q}(\Omega)$ the cases with $s \neq 0$ follow from the theorems before. The rest is a matter of interpolation.

For $s > \frac{1}{p}$ there are no interior $u$-wavelet systems for $A^{s}_{p,q}(\Omega)$ because of boundary values. So we have to find wavelet systems having values at the boundary and omit “interior”.


Traces on the boundary of cubes (i)

Let $Q = \{ x \in \mathbb{R}^n : x = (x_1, \ldots, x_n), 0 < x_m < 1, m = 1, \ldots, n \}$. The boundary $\Gamma = \partial Q$ of $Q$ can be represented as

$$\Gamma = \bigcup_{\ell=0}^{n-1} \Gamma_{\ell} \text{ with } \Gamma_{\ell} \cap \Gamma_{\ell'} = \emptyset \text{ for } \ell \neq \ell',$$

where $\Gamma_{\ell} = \bigcup_{j=0}^{N_{\ell}} Q_{\ell,j}$ consists of all $\ell$-dimensional faces $Q_{\ell,j}$ of $Q$, which are disjoint cubes of dimension $\ell$. 
Traces on the boundary of cubes (i)

Let $Q = \{ x \in \mathbb{R}^n : x = (x_1, \ldots, x_n), 0 < x_m < 1, m = 1, \ldots, n \}$. The boundary $\Gamma = \partial Q$ of $Q$ can be represented as

$$\Gamma = \bigcup_{\ell=0}^{n-1} \Gamma_\ell$$

with $\Gamma_\ell \cap \Gamma_{\ell'} = \emptyset$ for $\ell \neq \ell'$,

where $\Gamma_\ell = \bigcup_{j=0}^{N_\ell} Q_{\ell,j}$ consists of all $\ell$-dimensional faces $Q_{\ell,j}$ of $Q$, which are disjoint cubes of dimension $\ell$.

Let $\text{tr}_{\ell,j}$ be the restriction of $f \in A_{p,q}^s(\mathbb{R}^n)$ to $Q_{\ell,j}$ and

$$\text{tr}_{\ell,j}^r : f \mapsto \text{TR}_{\ell,j}^r(f) := \prod \{ \text{tr}_{\ell,j} D_\gamma^\alpha f : |\alpha| \leq r, j = 0, \ldots, N_\ell \},$$

where only derivatives perpendicular to $Q_{\ell,j}$ are admitted. Then we consider the composite mapping for $\tilde{r} = (r_{\ell_0}, \ldots, r_{n-1})$, $\ell_0 \leq n - 1$ and $r_\ell = \lfloor s - \frac{n-\ell}{p} \rfloor$.

$$\text{tr}_{\tilde{r}}^r : f \mapsto \prod_{\ell=\ell_0}^{n-1} \text{TR}_{\ell}^{r_\ell}(f).$$
Traces on the boundary of cubes (ii)

**Theorem**

Let

\[ 1 \leq p < \infty, 0 < q < \infty, s > \frac{1}{p} \text{ and } s - \frac{k}{p} \notin \mathbb{N}_0 \text{ for } k = 1, \ldots, n. \]

Let \( \ell_0 = 0 \) if \( s > \frac{n}{p} \). Otherwise \( \ell_0 \in \mathbb{N} \) is chosen such that

\[ 0 < s - \frac{n - \ell_0}{p} < \frac{1}{p}. \]

Then

\[ tr_{\Gamma}^{r} : B_{p,q}^{s}(Q) \mapsto \prod_{\ell=\ell_0}^{n-1} \prod_{j=0}^{N_\ell} \prod_{|\alpha| \leq r^\ell} B_{p,q}^{s-n-\ell} \left( Q_{\ell,j} \right), \]

\[ tr_{\Gamma}^{r} : F_{p,q}^{s}(Q) \mapsto \prod_{\ell=\ell_0}^{n-1} \prod_{j=0}^{N_\ell} \prod_{|\alpha| \leq r^\ell} B_{p,p}^{s-n-\ell} \left( Q_{\ell,j} \right). \]
Traces on the boundary of cubes (iii)

**Theorem**

Let

\[ 1 \leq p < \infty, 0 < q < \infty, s > \frac{1}{p} \quad \text{and} \quad s - \frac{k}{p} \notin \mathbb{N}_0 \quad \text{for} \quad k = 1, \ldots, n. \]

Let \( \ell_0 = 0 \) if \( s > \frac{n}{p} \). Otherwise \( \ell_0 \in \mathbb{N} \) is chosen such that

\[ 0 < s - \frac{n - \ell_0}{p} < \frac{1}{p}. \]

Then

\[ \tilde{B}_p^s(Q) = \left\{ f \in B_p^s(Q) : \text{tr}^{\tilde{r}}_\Gamma = 0 \right\}, \]

\[ \tilde{F}_p^s(Q) = \left\{ f \in F_p^s(Q) : \text{tr}^{\tilde{r}}_\Gamma = 0 \right\}. \]
Extension operators for the boundary of cubes (i)

**Theorem**

Let \( p, q, s, \ell_0 \) as in the theorem before and additionally \( s < u \).

Then there is a **wavelet-friendly** extension operator

\[
\begin{align*}
\text{Ext}_{\Gamma}^{r,u} : & \prod_{\ell=\ell_0}^{n-1} \prod_{j=0}^{N_{\ell}} \prod_{|\alpha| \leq r} \tilde{B}_{p,q}^{s-n-\ell-|\alpha|} (Q_{\ell,j}) \mapsto B_{p,q}^s (Q), \\
\text{Ext}_{\Gamma}^{r,u} : & \prod_{\ell=\ell_0}^{n-1} \prod_{j=0}^{N_{\ell}} \prod_{|\alpha| \leq r} \tilde{B}_{p,p}^{s-n-\ell-|\alpha|} (Q_{\ell,j}) \mapsto F_{p,q}^s (Q).
\end{align*}
\]

It holds

\[
tr_{\Gamma} \circ \text{ext}_{\Gamma}^{r,u} = id.
\]
Extension operators for the boundary of cubes (ii)

**Theorem**

Furthermore,

\[
B_{p,q}^s(Q) = \tilde{B}_{p,q}^s(Q) \times \text{Ext}_{\Gamma}^{\tilde{r},u} \prod_{\ell=\ell_0}^{n-1} \prod_{j=0}^{N_\ell} \prod_{|\alpha| \leq r^\ell} \tilde{B}_{p,q}^{s-n-\ell} - |\alpha| (Q_{\ell,j}),
\]

\[
F_{p,q}^s(Q) = \tilde{F}_{p,q}^s(Q) \times \text{Ext}_{\Gamma}^{\tilde{r},u} \prod_{\ell=\ell_0}^{n-1} \prod_{j=0}^{N_\ell} \prod_{|\alpha| \leq r^\ell} \tilde{B}_{p,p}^{s-n-\ell} - |\alpha| (Q_{\ell,j}).
\]
**Theorem**

Furthermore,

\[
B_{p,q}^s(Q) = \tilde{B}_{p,q}^s(Q) \times \text{Ext}_{\Gamma, u}^{\bar{r}, n-1} \prod_{\ell = \ell_0}^{n-1} \prod_{j = 0}^{N_{\ell}} \prod_{|\alpha| \leq r_{\ell}} \tilde{B}_{p,q}^{s-n-\ell p-|\alpha|} (Q_{\ell,j}),
\]

\[
F_{p,q}^s(Q) = \tilde{F}_{p,q}^s(Q) \times \text{Ext}_{\Gamma, u}^{\bar{r}, n} \prod_{\ell = \ell_0}^{n-1} \prod_{j = 0}^{N_{\ell}} \prod_{|\alpha| \leq r_{\ell}} \tilde{B}_{p,p}^{s-n-\ell p-|\alpha|} (Q_{\ell,j}).
\]

Proof (sketch): Take first the traces to the faces of dimension \(\ell_0\), then extend it back to \(B_{p,q}^s(Q)\). These maps are projections. The complementary space is \(\{ f \in B_{p,q}^s(Q) : \text{tr} f = 0 \}\). Now take the trace of this space to the faces of dimension \(\ell = \ell_0 + 1\) of this space. By boundary considerations this is \(\tilde{B}_{p,q}^{s-n-\ell} (Q_{\ell,j})\). And so on.
Wavelets for cubes

**Theorem (T08 - Theorem 6.30)**

Let $A^s_{p,q}(Q)$ (can be extended to cellular domains) be given with

$$1 \leq p < \infty, s > \frac{1}{p} \text{ and } s - \frac{k}{p} \notin \mathbb{N}_0 \text{ for } k = 1, \ldots, n, 0 < q < \infty$$

$q \geq 1$ for the F-spaces. Then there is an oscillating $u$-wavelet system with $u > s$ which is a Riesz basis in $A^s_{p,q}(Q)$. 
Theorem (T08 - Theorem 6.30)

Let $A^s_{p,q}(Q)$ (can be extended to cellular domains) be given with

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(q ≥ 1 for the F-spaces). Then there is an oscillating $u$-wavelet system with $u > s$ which is a Riesz basis in $A^s_{p,q}(Q)$.

Proof: Use the decomposition

$$B^s_{p,q}(Q) = \tilde{B}^s_{p,q}(Q) \times \text{Ext}_{\Gamma}^{\tilde{r},u} \prod_{\ell=\ell_0}^{n-1} \prod_{j=0}^{N_{\ell}} \prod_{|\alpha| \leq r^\ell} \tilde{B}^{s - \frac{n-\ell}{p} - |\alpha|}_{p, q}(Q_{\ell,j})$$

and the fact that every space on the right hand side has a $u$-wavelet system which is a Riesz basis. Now one must ensure that wavelet-friendly extension operators are really wavelet-friendly.
The case \( s - \frac{1}{p} \in \mathbb{N}_0 \)

Let \( \Omega \) be now a \( C^\infty \)-domain (so also a cellular domain) and let \( 1 < p, q < \infty \). By a translation and localization argument one always has

\[
\tilde{A}^s_{p,q}(\Omega) \hookrightarrow \hat{A}^s_{p,q}(\Omega).
\]

The converse is true if \( s - \frac{1}{p} \notin \mathbb{N} \). Using the Hardy inequalities from [T01] one gets

\[
\int_{\Omega} \frac{|f(x)|^p}{d^{sp}(x)} \, dx \leq c \|f\|_{F^s_{p,q}(\mathbb{R}^n)} \text{ for } f \in \tilde{F}^s_{p,q}(\Omega),
\]

where \( d(x) = dist(x, \Gamma) \). Then \( \chi_\Omega \notin \tilde{F}^s_{p,q}(\Omega) \) for \( s \geq \frac{1}{p} \).
The case $s - \frac{1}{p} \in \mathbb{N}_0$

Let $\Omega$ be now a $C^\infty$-domain (so also a cellular domain) and let $1 < p, q < \infty$. By a translation and localization argument one always has

$$\tilde{A}^s_{p,q}(\Omega) \hookrightarrow \hat{A}^s_{p,q}(\Omega).$$

The converse is true if $s - \frac{1}{p} \notin \mathbb{N}$. Using the Hardy inequalities from [T01] one gets

$$\int_{\Omega} \frac{|f(x)|^p}{d^{sp}(x)} \, dx \leq c \|f|F^s_{p,q}(\mathbb{R}^n)\| \quad \text{for } f \in \tilde{F}^s_{p,q}(\Omega),$$

where $d(x) = dist(x, \Gamma)$. Then $\chi_{\Omega} \notin \tilde{F}^s_{p,q}(\Omega)$ for $s \geq \frac{1}{p}$.

On the other hand $\chi_{\Omega} \in \check{F}^1_{p,q}(\Omega) = F^1_{p,q}(\Omega)$ by an atomic decomposition argument.

Analogously, it follows $\tilde{F}^s_{p,q}(\Omega) \neq \check{F}^s_{p,q}(\Omega)$ if $s - \frac{1}{p} \in \mathbb{N}$. There are no Riesz frames for these $\tilde{F}^s_{p,q}(\Omega)$ (T08 - Prop. 6.40.).
The cases \( s - \frac{k}{p} \in \mathbb{N}_0 \) with \( k = 2, \ldots, n \)

It is known that not all the conditions are necessary depending on the kind of domain.

- For planar \( C^\infty \)-domains \((n = 2)\) the restriction for \( k = 2 \) is not necessary,
- For \( C^\infty \)-domains with boundaries diffeomorphic to \( S^n \) the condition for \( k = n \) is not necessary,
- For arbitrary \( C^\infty \)-domains and

\[
1 \leq p < \infty, \quad \frac{1}{p} - 1 < s < \frac{2}{p}, \quad s \neq \frac{1}{p}
\]

one finds a Riesz u-wavelet basis without further restrictions since the boundary has a basis,

- For the torus one needs no further conditions since the boundary is equipped with a u-wavelet basis ([T08] - sect. 1.3.2). So \( W^1_2(\Omega) \) has a basis if \( \Omega \) is a torus which is unknown if \( \Omega \) is a ball and \( n \geq 3 \).
Reinforced spaces

Let $\Omega_\varepsilon = \{ x \in \Omega, d(x) < \varepsilon \}$ and

$$F_{p,q}^{s,\text{rinf}}(\Omega) = \begin{cases} F_{p,q}^s(\Omega) & \text{if } s - \frac{1}{p} \notin \mathbb{N}_0 \\ f \in F_{p,q}^s(\Omega) : d^{\frac{1}{p}} \frac{\partial f}{\partial \nu^r} \in L_p(\Omega_\varepsilon) & \text{if } s - \frac{1}{p} = r \in \mathbb{N}_0. \end{cases}$$

If $\Omega$ is a bounded $C^\infty$-domain, then

$$\tilde{F}_{p,q}^s(\Omega) = \left\{ f \in F_{p,q}^{s,\text{rinf}}(\Omega) : \text{tr}_{\Gamma}^{r-1} f = 0 \right\}.$$
Reinforced spaces

Let $\Omega_\varepsilon = \{ x \in \Omega, d(x) < \varepsilon \}$ and

$$F_{p,q}^{s,r_{\text{inf}}} (\Omega) = \begin{cases} F_{p,q}^s (\Omega) & \text{if } s - \frac{1}{p} \notin \mathbb{N}_0 \\ f \in F_{p,q}^s (\Omega) : d^{-\frac{1}{p}} \frac{\partial f}{\partial \nu} \in L_p(\Omega_\varepsilon) & \text{if } s - \frac{1}{p} = r \in \mathbb{N}_0. \end{cases}$$

If $\Omega$ is a bounded $C^\infty$-domain, then

$$\tilde{F}_{p,q}^s (\Omega) = \left\{ f \in F_{p,q}^{s,r_{\text{inf}}} (\Omega) : \text{tr}_r^{-1} f = 0 \right\}.$$

This and the observation that the known extension operators also map to $F_{p,q}^{s,r_{\text{inf}}} (\Omega)$ instead of $F_{p,q}^s (\Omega)$ is the starting point for finding a u-wavelet basis.
Reinforced spaces advanced

Roughly speaking: This gives u-wavelet bases for the modified spaces $F_{p,q}^{s,rinf}(\Omega)$ in the cases of domains where no more restrictions than $s - \frac{1}{p} \not\in \mathbb{N}_0$ are necessary.
Roughly speaking: This gives \( u \)-wavelet bases for the modified spaces \( F_{s,r_{\text{inf}}}^{p,q}(\Omega) \) in the cases of domains where no more restrictions than \( s - \frac{1}{p} \notin \mathbb{N}_0 \) are necessary.

The big question: Can one extend this idea to cellular domains with more restrictions?
Reinforced spaces advanced

Roughly speaking: This gives u-wavelet bases for the modified spaces $F_{p,q}^{s,rinf}(\Omega)$ in the cases of domains where no more restrictions than $s - \frac{1}{p} \notin \mathbb{N}_0$ are necessary.

The big question: Can one extend this idea to cellular domains with more restrictions?

The problem: One has to ensure a composition of type.

$$\tilde{F}_{p,q}^{s}(\Omega) = \left\{ f \in F_{p,q}^{s,rinf}(\Omega) : tr_{r-1} \Gamma f = 0 \right\}.$$ 

and has to check if the extension operators are suitable (which they should be). Furthermore, we need conditions for defining $F_{p,q}^{s,rinf}(\Omega)$ which handle the lower dimensional boundary faces. This has not been done so far. In [T08], sect. 6.2.4, there is some overview given for $W_2^1(Q)$.
Thank you for your attention

Questions?