Equivalent norms and characterizations for vector-valued function spaces

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Oppurg, October 15, 2010

Guideline of the talk

Function Spaces $B^s_{p,q}(E)$, $F^s_{p,q}(E)$ of functions

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where E is a Banach space.

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$$f = \sum_{\nu=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{\nu,m} a_{\nu,m}.$$ Not today!

The spaces $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n, E)$

Let $\mathcal{S}(\mathbb{R}^n)$ be the set of the infinitely often differentiable functions $\varphi : \mathbb{R}^n \to \mathbb{C}$ for which the seminorms

$$\|\varphi\|_{K,L} := \sup_{x\in\mathbb{R}^n} (1 + |x|^2)^{\frac{K}{2}} \sum_{|\alpha|\leq L} |D^\alpha \varphi(x)|$$

for $K, L \in \mathbb{N}_0$ are finite.

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We say that a linear map $f : \mathcal{S}(\mathbb{R}^n) \to E$ is an E-valued tempered distribution if there exist a constant $c > 0$ and $K, L \in \mathbb{N}_0$ such that

$$\|f(\varphi)|E\| \leq c\|\varphi\|_{K,L}$$

for all $\varphi \in \mathcal{S}(\mathbb{R}^n)$. The set of all this linear mappings equipped with the weak topology is denoted by $\mathcal{S}'(\mathbb{R}^n, E)$.

Bochner integral of vector-valued functions

Let $0 < p \leq \infty$. We define

$$L_p(E) := \left\{ f : \mathbb{R}^n \to E, f \text{ measurable } , \left\| \|f|E\| \middle| L_p(\mathbb{R}^n, \mathbb{C}) \right\| < \infty \right\}.$$

Warning! Unusual use of letter E in brackets.

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For

$$f = \sum_{m=1}^{M} a_i \cdot e_i$$

with $e_i \in E$ and $a_i \in L_1(\mathbb{C})$ we set

$$\int_{\mathbb{R}^n} f(x)\, dx := \sum_{m=1}^{M} e_i \cdot \int_{\mathbb{R}^n} a_i(x)\, dx \in E.$$

Bochner integral of vector-valued functions (ii)

It holds:

- The integral is independent of the representation.
- The subspace of functions of this type is dense in $L_1(E)$.
- We have the estimate

$$\left\| \int_{\mathbb{R}^n} f(x)\ dx \middle| E \right\| \le \left\| \|f|E\| \middle| L_1(\mathbb{C}) \right\| = \|f|L_1(E)\|.$$

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$\Rightarrow$ There is a unique continuation: A continuous linear operator from $L_1(E)$ to E with norm 1 called Bochner integral.

$\Rightarrow$ For a linear functional $a : E \to \mathbb{C}$ one has

$$a\left(\int_{\mathbb{R}^n} f(x)\, dx\right) = \int_{\mathbb{R}^n} a(f(x))\, dx.$$

Fourier transform of $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n, E)$

The Fourier transform $\hat{\varphi}$ of $\varphi \in \mathcal{S}(\mathbb{R}^n)$ is given by

$$\hat{\varphi}(\xi) := (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \varphi(x) e^{-ix\xi}\, dx,$$

analogously its inverse $\check{\varphi}$. We have $\hat{f} \in \mathcal{S}(\mathbb{R}^n)$ iff $f \in \mathcal{S}(\mathbb{R}^n)$.

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The Fourier transform $\hat{f}$ for $f \in \mathcal{S}'(\mathbb{R}^n, E)$ is given by

$$\hat{f}(\varphi) := f(\hat{\varphi}) \text{ for } \varphi \in \mathcal{S}(\mathbb{R}^n),$$

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Fourier transform for $L_1(E)$ and convolution

If $f \in L_p(E)$ for a $p \in [1, \infty]$, then we can think of f as an element of $\mathcal{S}'(\mathbb{R}^n, E)$ by defining

$$f(\varphi) := \int_{\mathbb{R}^n} \underbrace{f(x)}_{\in E} \underbrace{\varphi(x)}_{\in \mathbb{C}} \; dx.$$

Distributions of such an integral form are called regular.

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As in the scalar case we have: If $f \in L_1(E)$, then $\hat{f}$ is a regular distribution, bounded and it holds

$$\hat{\varphi}(\xi) := (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \varphi(x) e^{-ix\xi} \, dx.$$

For $f \in \mathcal{S}'(\mathbb{R}^n, E)$ and $\varphi \in \mathcal{S}(\mathbb{R}^n)$ we define the convolution of f and φ as the function

$$(f * \varphi)(x) := (2\pi)^{-\frac{n}{2}} \cdot f(\varphi(x - \cdot)) \text{ for all } x \in \mathbb{R}^n.$$

The definition of $B^s_{p,q}(E)$

Let $\{\varphi_j\}_{j=0}^{\infty}$ be a resolution of unity. Let $0 < p \leq \infty$, $0 < q \leq \infty$ and $s \in \mathbb{R}$. For $f \in \mathcal{S}'(\mathbb{R}^n, E)$ we define

$$\|f|B^s_{p,q}(E)\|^{\varphi} := \left(\sum_{j=0}^{\infty} 2^{jsq} \|(\varphi_j \hat{f})^{\vee}|L_p(E)\|^q \right)^{\frac{1}{q}}$$

(modified in case $q = \infty$) and

$$B^{s,\varphi}_{p,q}(E) := \left\{ f \in \mathcal{S}'(\mathbb{R}^n, E) : \|f|B^s_{p,q}(E)\|^{\varphi} < \infty \right\}.$$

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$$B^{s,\varphi}_{p,q}(E) := \left\{ f \in \mathcal{S}'(\mathbb{R}^n, E) : \|f|B^s_{p,q}(E)\|^\varphi < \infty \right\}.$$

Then $(B^{s,\varphi}_{p,q}(E), \|\cdot|B^s_{p,q}\|^\varphi(E))$ is a quasi-Banach space. It does not depend on the choice of the resolution of unity $\{\varphi_j\}_{j=0}^\infty$ in the sense of equivalent norms. So we denote it shortly by $B^s_{p,q}(E)$.

A deeper look at the resolution of unity

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- at first $\varphi_0, \varphi \in \mathcal{S}(\mathbb{R}^n)$ to define a convolution
- Fourier multiplier theorems for functions whose Fourier transform have compact support:

$$supp\ \varphi_j \hat{f} \subset supp\ \varphi_j \subset \left\{2^{j-1} \leq |x| \leq 2^{j+1}\right\},$$

- the property $|D^\alpha \varphi_j(x)| \leq c_\alpha 2^{-j|\alpha|}$
- the unit property $\sum_{j=0}^{\infty} \varphi_j(x) = 1$.

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- the unit property $\sum_{j=0}^{\infty} \varphi_j(x) = 1$.

Question: Is there a possibility to weaken the conditions on φ_0, φ?

Triebel's approach for the scalar case

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- not only discrete but also continuous norms

$$\left(\|(\varphi_0 \hat{f})^\vee | L_p\| + \sum_{j=1}^\infty 2^{jsq} \|(\varphi(2^{-j}\cdot)\hat{f})^\vee | L_p\|^q \right)^{\frac{1}{q}}$$

$$\Rightarrow \|(\varphi_0 \hat{f})^\vee | L_p\| + \left(\int_0^1 t^{-sq} \|(\varphi(t\cdot)\hat{f})^\vee | L_p\|^q \, \frac{dt}{t} \right)^{\frac{1}{q}}$$

Rychkov's approach for the scalar case

V. S. Rychkov: On a Theorem of Bui, Paluszyński, and Taibleson. Proc. Steklov Inst. Math. 227, 280, 1999

Let $S \in \mathbb{Z}$ *with* $S \geq \lfloor s \rfloor$, $\varphi_0, \varphi \in \mathcal{S}(\mathbb{R}^n)$ *and* $\varepsilon > 0$ *such that*

$$|\varphi_0(x)| > 0 \text{ for } \{|x| < 2\varepsilon\}, \quad |\varphi(x)| > 0 \text{ for } \left\{\frac{\varepsilon}{2} < |x| < 2\varepsilon\right\},$$

$$D^\alpha \varphi(0) = 0 \text{ for } |\alpha| \leq S.$$

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Let $\varphi_j := \varphi(2^{-j}\cdot)$. *Then*

$$\|f|B^s_{p,q}\|_{\varphi_0,\varphi} := \|(\varphi_0 \hat{f})^\vee|L_p\| + \left(\sum_{j=1}^{\infty} 2^{jsq}\|(\varphi_j \hat{f})^\vee|L_p\|^q\right)^{\frac{1}{q}}$$

is an equivalent norm for $B^s_{p,q}$. *It holds*

$$B^s_{p,q} = \left\{f \in \mathcal{S}'(\mathbb{R}^n) : \|f|B^s_{p,q}\|_{\varphi_0,\varphi} < \infty\right\}.$$

Comparison of the two theorems/proofs

Advantages of the Triebel proof

- more general result (characterization by differences)
- no need of $\mathcal{S}(\mathbb{R}^n)$ functions
- continuous norms

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Advantages of the Rychkov proof

- much easier proof
- fewer conditions on φ, for $s < 0$ even no more conditions (this does not come out of Triebel's proof)
- gives characterizations for all s, p, q, not only equivalent norms

Rychkov's approach for discrete vector-valued norms

Theorem

Let $S \in \mathbb{Z}$ *with* $S \geq \lfloor s \rfloor$, φ_0, $\varphi \in \mathcal{S}(\mathbb{R}^n)$ *and* $\varepsilon > 0$ *such that*

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Let $\varphi_j := \varphi(2^{-j}\cdot)$. *Then*

$$\|f|B^s_{p,q}(E)\|^\varphi := \|(\varphi_0 \hat{f})^\vee|L_p(E)\| + \left(\sum_{j=1}^{\infty} 2^{jsq} \|(\varphi_j \hat{f})^\vee|L_p(E)\|^q\right)^{\frac{1}{q}}$$

is an equivalent norm for $B^s_{p,q}(E)$. *It holds*

$$B^s_{p,q}(E) = \left\{f \in \mathcal{S}'(\mathbb{R}^n, E) : \|f|B^s_{p,q}(E)\|^\varphi < \infty\right\}.$$

Rychkov's approach for continuous vector-valued norms

Theorem

Let $S \in \mathbb{Z}$ *with* $S \geq \lfloor s \rfloor$, $\varphi_0, \varphi \in \mathcal{S}(\mathbb{R}^n)$ *and* $\varepsilon > 0$ *such that*

$$|\varphi_0(x)| > 0 \text{ for } \{|x| < 2\varepsilon\}, \quad |\varphi(x)| > 0 \text{ for } \left\{\frac{\varepsilon}{2} < |x| < 2\varepsilon\right\},$$
$$D^\alpha \varphi(0) = 0 \text{ for } |\alpha| \leq S.$$

Then

$$\|f|B^s_{p,q}(E)\|^I = \|(\varphi_0 \hat{f})^\vee|L_p(E)\| + \left(\int_0^1 t^{-sq} \|(\varphi(t\cdot)\hat{f})^\vee|L_p(E)\|^q \frac{dt}{t}\right)^{\frac{1}{q}}$$

is an equivalent norm for $B^s_{p,q}(E)$. *It holds*

$$B^s_{p,q}(E) = \left\{f \in \mathcal{S}'(\mathbb{R}^n, E) : \|f|B^s_{p,q}(E)\|^I < \infty\right\}.$$

Some improvement is possible

- One can replace $(\varphi_j \hat{f})^\vee$ by

$$(\varphi_j^* f)_a(x) := \sup_{y \in \mathbb{R}^n} \frac{\|(\varphi(2^{-j}\cdot)\hat{f})^\vee(x-y)|E\|}{(1+2^j|y|)^a},$$

where a must be bigger than $\frac{n}{p}$ and additionally bigger than $\frac{n}{q}$ for $F^s_{p,q}(E)$.
(will be called maximal function norm $\|\cdot|B^s_{p,q}(E)\|^*$)

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- Let $d > 0$ be given. One can replace $\|(\varphi(t\cdot)\hat{f})\check{}(x)|E\|$ by

$$\sup \|(\varphi(\tau\cdot)\hat{f})\check{}(y)|E\|,$$

where sup is the supremum over

$$\{|x-y| \leq dt, t \leq \tau \leq 2t\}.$$

A remark on the proof (i)

Consists of two main parts:

1^{st} step: Show that for two different φ_0's and φ's with the necessary conditions the two maximal function norms $\| \cdot | B^s_{p,q}(E)\|^*$ are equivalent.

A remark on the proof (i)

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1^{st} step: Show that for two different φ_0's and φ's with the necessary conditions the two maximal function norms $\| \cdot | B^s_{p,q}(E)\|^*$ are equivalent.

2^{nd} step: Show that for fixed φ the maximal function norm $\| \cdot | B^s_{p,q}(E)\|^*$ can be estimated from above by $\| \cdot | B^s_{p,q}(E)\|^\varphi$. (the converse estimate is trivial:

$$\|(\varphi_j \hat{f})^\vee(x)|E\| \leq \sup_{y \in \mathbb{R}^n} \frac{\|(\varphi(2^{-j}\cdot)\hat{f})^\vee(x-y)|E\|}{(1+2^j|y|)^a}.)$$

A remark on the proof (ii)

The most important difference to Triebel's proof is the given characterization of $B^s_{p,q}(E)$. But there is/was one problem in the 2^{nd} step of Rychkov's proof (of the scalar case): One has to estimate

$$d_j = (\varphi_j^* f)_a(x) = \sup_{y \in \mathbb{R}^n} \frac{\|(\varphi(2^{-j}\cdot)\hat{f})^\vee(x-y)|E\|}{(1+2^j|y|)^a} = \sup_{y \in \mathbb{R}^n} \frac{\|\left(\check{\varphi}_j * f\right)(x-y)|E\|}{(1+2^j|y|)^a}$$

from above by $c \cdot 2^{jN}$, where N is an arbitrary natural number and c is independent of j.

A remark on the proof (iii)

This can be done if we assume a priori $f \in B^s_{p,q}(E)$ or $F^s_{p,q}(E)$: By the lift property and the Sobolev embeddings there is a $\sigma \in \mathbb{N}$ such that $g := ((1+|\xi|^2)^{-\sigma}\hat{f})^\vee \in L_\infty(E)$.

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$$
\begin{aligned}
(\varphi_j^* f)_a(x) &= \sup_{y\in\mathbb{R}^n} \frac{\left\| \left(\varphi(2^{-j}\cdot)(1+|\xi|^2)^{+\sigma}(1+|\xi|^2)^{-\sigma}\hat{f}\right)^\vee (y) \middle| E \right\|}{(1+2^j|x-y|)^a} \\
&= \sup_{y\in\mathbb{R}^n} \frac{\left\| \left((\varphi(2^{-j}\cdot)(1+|\xi|^2)^{\sigma})^\vee * g\right)(y) \middle| E \right\|}{(1+2^j|x-y|)^a} \\
&\le \|g|L_\infty(E)\| \cdot \|(\varphi(2^{-j}\cdot)(1+|\xi|^2)^{\sigma})^\vee|L_1\| \\
&\le C\|g|L_\infty(E)\| \cdot \sum_{|\alpha|\le 2\sigma} \|2^{jn}D^\alpha\check{\varphi}(2^j\cdot)|L_1\| \\
&\le C\cdot 2^{2j\sigma}\|g|L_\infty(E)\| \cdot \sum_{|\alpha|\le 2\sigma} \|D^\alpha\check{\varphi}|L_1\|.
\end{aligned}
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A remark on the proof (iv)

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&\leq \sup_{y\in\mathbb{R}^n} \frac{(1+|x-y|^2)^{\frac{K}{2}}}{(1+2^j|y|)^a} \|2^{jn}\check{\varphi}(2^j\cdot)\|_{K,L} \\
&\leq c\,(1+|x|^2)^{\frac{K}{2}} 2^{j(L+n)} \|\check{\varphi}\|_{K,L}
\end{aligned}
$$

if $a \geq K$ and K is a constant depending on the norm of f as an element in $\mathcal{S}'(\mathbb{R}^n, E)$

$$
\|\varphi\|_{K,L} := \sup_{y\in\mathbb{R}^n} (1+|y|^2)^{\frac{K}{2}} \sum_{|\alpha|\leq L} \|D^\alpha \varphi(y)\|.
$$

A remark on the proof (v)

So here is how the crucial part of the proof proceeds:

1. Let $f \in \mathcal{S}'(\mathbb{R}^n, E)$ be given with finite norm $\|f|B^s_{p,q}(E)\|^\varphi$ for an admissible φ.

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2. By the 2^{nd} step the maximal function norm $\|f|B^s_{p,q}(E)\|^*$ can be estimated from above by $\|f|B^s_{p,q}(E)\|^\varphi$ if $a > K$ (and K depends on f) and hence is finite as well.

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3. By the 1^{st} step the function is in $B^s_{p,q}(E)$ (see the definition of $B^s_{p,q}(E)$).
4. Then we have the necessary estimate of $d_j = (\varphi_j^* f)_a(x)$ for all admissible a and hence the 2^{nd} step can be applied once more for all $a > \frac{n}{p}$.

Vector-valued local means

Let $k_0, k^0 \in \mathcal{S}(\mathbb{R}^n)$, $\hat{k_0}(0) \neq 0$, $\hat{k^0}(0) \neq 0$, $N \in \mathbb{N}_0$ with $2N > s$ and $k^N := \Delta^N k^0$.
Then $\varphi_0 := \hat{k_0}$ und $\varphi := \widehat{k^N}$ fulfil the conditions of the theorem.
In particular, k_0 and k^0 can be chosen with support in $\{|x| \leq 1\}$.

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In particular, k_0 and k^0 can be chosen with support in $\{|x| \leq 1\}$.
Let $k_j^N = 2^{jn} k^N(2^j \cdot)$.

Theorem (local means)

Let $2N > s$, $0 < p \leq \infty$ and $0 < q \leq \infty$. Then

$$\|f|B^s_{p,q}(E)\|_{k_0,k^N} := \|k_0 * f|L_p(E)\| + \left(\sum_{j=1}^{\infty} 2^{jsq} \|k_j^N * f|L_p(E)\|^q \right)^{\frac{1}{q}}$$

is an equivalent norm for $B^s_{p,q}(E)$. It holds

$$B^s_{p,q}(E) = \left\{ f \in \mathcal{S}'(\mathbb{R}^n, E) : \|f|B^s_{p,q}(E)\|_{k_0,k^N} < \infty \right\}.$$

The end

Thank you for your attention

Questions?